Fluid Models, with Applications to Staffing

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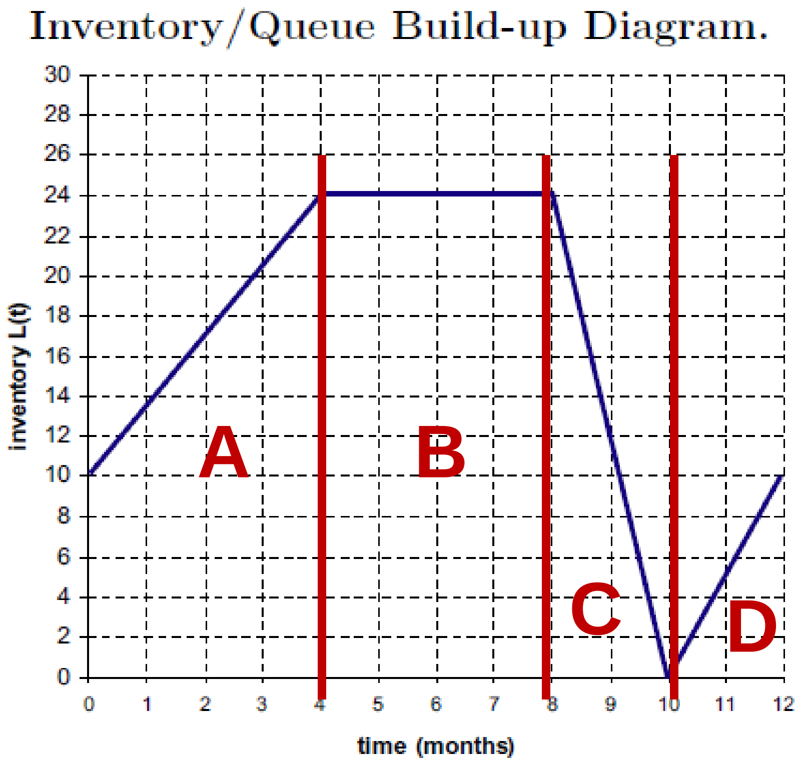
Part 1. Queue Build-Up Diagram

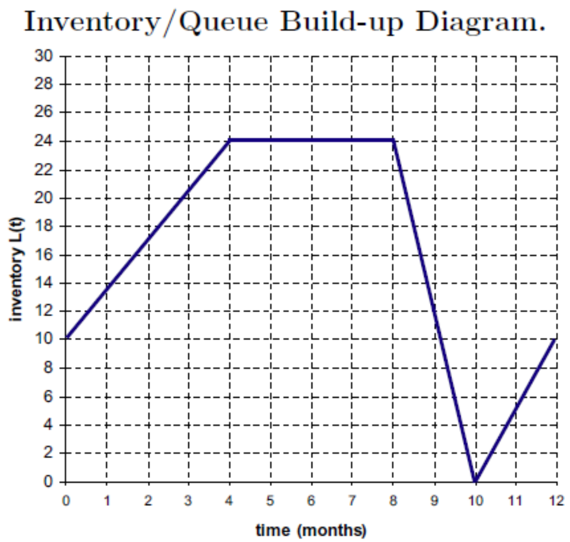
- **Process Flow:** A supermarket receives from suppliers 25 tons of fish per month. The average quantity of fish held in freezer storage is *16.5 tons*. The amount sold is 21.5 (tons per month) in Jan-Apr, 25 in May-Aug, 37 in Sep-Oct and 20 in Nov-Dec. There was 10 tons of inventory at the beginning.



A: Queue build-up
B: Queue constant
C: Queue decreases
D: Queue build-up

Part 1. Queue Build-Up Diagram (2)

- On average, how long does a ton of fish remain in freezer storage between the time it is received and the time it is sent to the sales department?
- We want to use the **Little's Law**. How do we compute **L**?
  - by calculating the area below the inventory build-up graph:



$$17 \times \frac{4}{12} + 24 \times \frac{4}{12} + 12 \times \frac{2}{12} + 5 \times \frac{2}{12} =$$
$$\frac{17}{3} + 8 + 2 + \frac{5}{6} = 16.5.$$

- Then, $W = \frac{L}{\lambda} = \frac{16.5}{25} = 0.66\ months$, on average, is the period that a ton of fish spends in the freezer.

Part 2. Fluid Models

- Deterministic View
  - $\boldsymbol{\lambda(t)}$ - instantaneous arrival rate at time t
  - $\boldsymbol{c(t)}$ - instantaneous capacity of the system, at time t (maximal potential processing rate)
  - $\boldsymbol{\delta(t)}$ - instantaneous processing rate at time t.
    - $\delta(t) \leq c(t)$.
  - $\boldsymbol{Q(t)}$ - total amount of material in the system, (being processed + queued) at time t.

Part 2. Fluid Models (2)

- Assume that $Q(0)$, $\lambda(t)$ and $\delta(t)$ are given for all $t \in [0, T]$.

- Then $\boldsymbol{Q(t)}$ is the solution of the nonlinear differential equation $\frac{d}{dt}Q(t) = \lambda(t) - \delta(t)$, $Q(0) = Q_0, t \in [0, T]$.

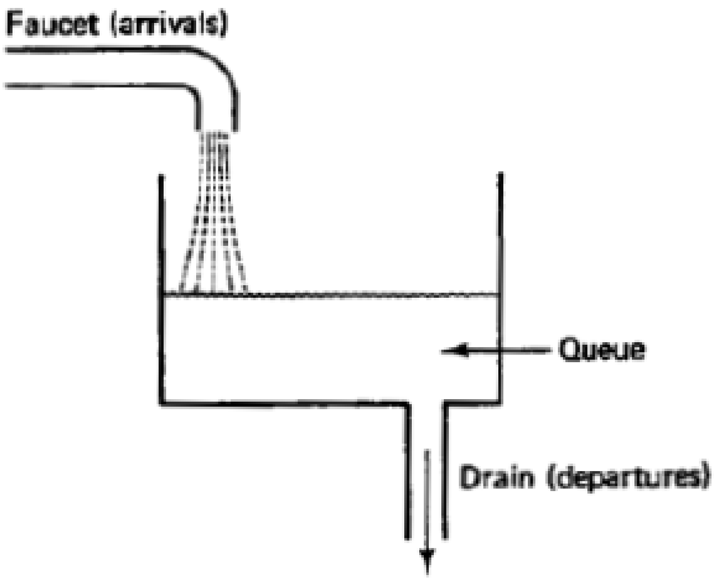


Figure 6.5 In a fluid model, the customers can be viewed as a liquid that accumulates in a tub. Queues increase when the fluid enters the tub faster than it leaves.

Part 2. Fluid Models (3)

- $\frac{d}{dt}Q(t) = \lambda(t) - \delta(t), Q(0) = Q_0, t \in [0, T]$
  - The general solution may be very complicated.
  - How to create or plot $Q(t) = (Q(0), Q(t_1), Q(t_2), \ldots, Q(T))$?

- Start with $Q(0)$. Then for $t_n = t_{n-1} + \Delta t, n = 1,2, \ldots$

$$\boldsymbol{Q(t_n) = Q(t_{n-1}) + \lambda(t_{n-1}) \cdot \Delta t - \delta(t_{n-1}) \cdot \Delta t}$$

- EXCEL software can be used (will be shown later in an example)

Part 2. Fluid Models (4)

- Each little tube (on slide 5) can be viewed as a server. Consider a queueing system with one class of customers and one service station.
  - $\lambda(t)$ - instantaneous arrival rate at time t
  - μ – service rate, constant in time
  - N(t) - # servers in the system at time t
  - $Q(t)$ - total # customers in the system, (being served + queued) at time t

  → $c(t) = \mu \cdot N(t)$

  → $\delta(t) = \mu \cdot (Q(t) \wedge N(t))$ (where $a \wedge b = \min(a, b)$)

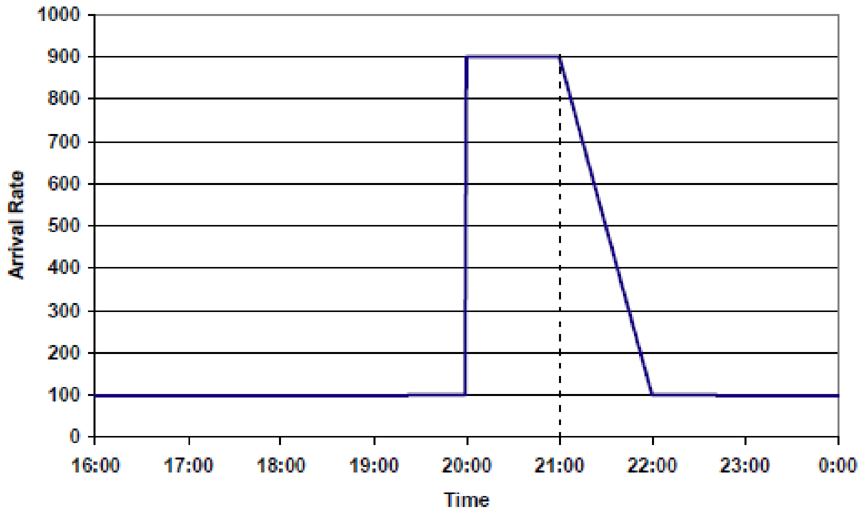
  → $\boldsymbol{\frac{d}{dt} Q(t) = \lambda(t) - \mu \cdot \big(Q(t) \wedge N(t)\big)}$

- If we add the possibility to abandon queue (where θ is the abandonment rate for each customer in queue):

  → $\boldsymbol{\frac{d}{dt} Q(t) = \lambda(t) - \mu \cdot \big(Q(t) \wedge N(t)\big) - \theta \cdot \big(Q(t) - N(t)\big)^{+}}$

EX. Tele-SHOP (HW Question)

- **Tele-SHOP** is a commercial channel dealing with online sales, which is operated by a call-center. In order to increase profits decided to place a 30-sec daily advertisement on national TV, at 20:00.
- It turns out that the arrivals to the call-center are well approximated by an inhomogeneous Poisson process, with an **arrival rate function** given by the graph:



- The call center operates from 16:00-24:00.
- The service duration of each incoming call has an average of 12 minutes.
- The number of servers in the system is fixed and equals N.

EX. Q(t)

- Assuming that no abandonment takes place, plot Q(t) for N=0,150,200.
  - We use an EXCEL spreadsheet

$$\boldsymbol{Q(6) = Q(5) + \lambda(5) \cdot \Delta t - \mu \cdot \big(Q(5) \wedge N(5)\big) \cdot \Delta t}$$

	A	B	C	D	E	F	G	H	I	J
1	Time	minute	lambda	mu	N	Number-in	Staffing Cost	Queue	Waiting Cost	Cost
2				0.08	50	System	0.8		1	174343.43
3	16:00	0	1.666667		50	0.000		0		
4	16:01	1	1.666667		50	1.667	40	0	0	
5	16:02	2	1.666667		50	3.194	40	0	0	
6	16:03	3	1.666667		50	4.595	40	0	0	
7	16:04	4	1.666667		50	5.879	40	0	0	
8	16:05	5	1.666667		50	7.055	40	0	0	
9	16:06	6	1.666667		50	8.134	40	0	0	
10	16:07	7	1.666667		50	9.123	40	0	0	
11	16:08	8	1.666667		50	10.029	40	0	0	
12	16:09	9	1.666667		50	10.860	40	0	0	
13	16:10	10	1.666667		50	11.622	40	0	0	

=100/6

=$E$2

=$G$2*E13

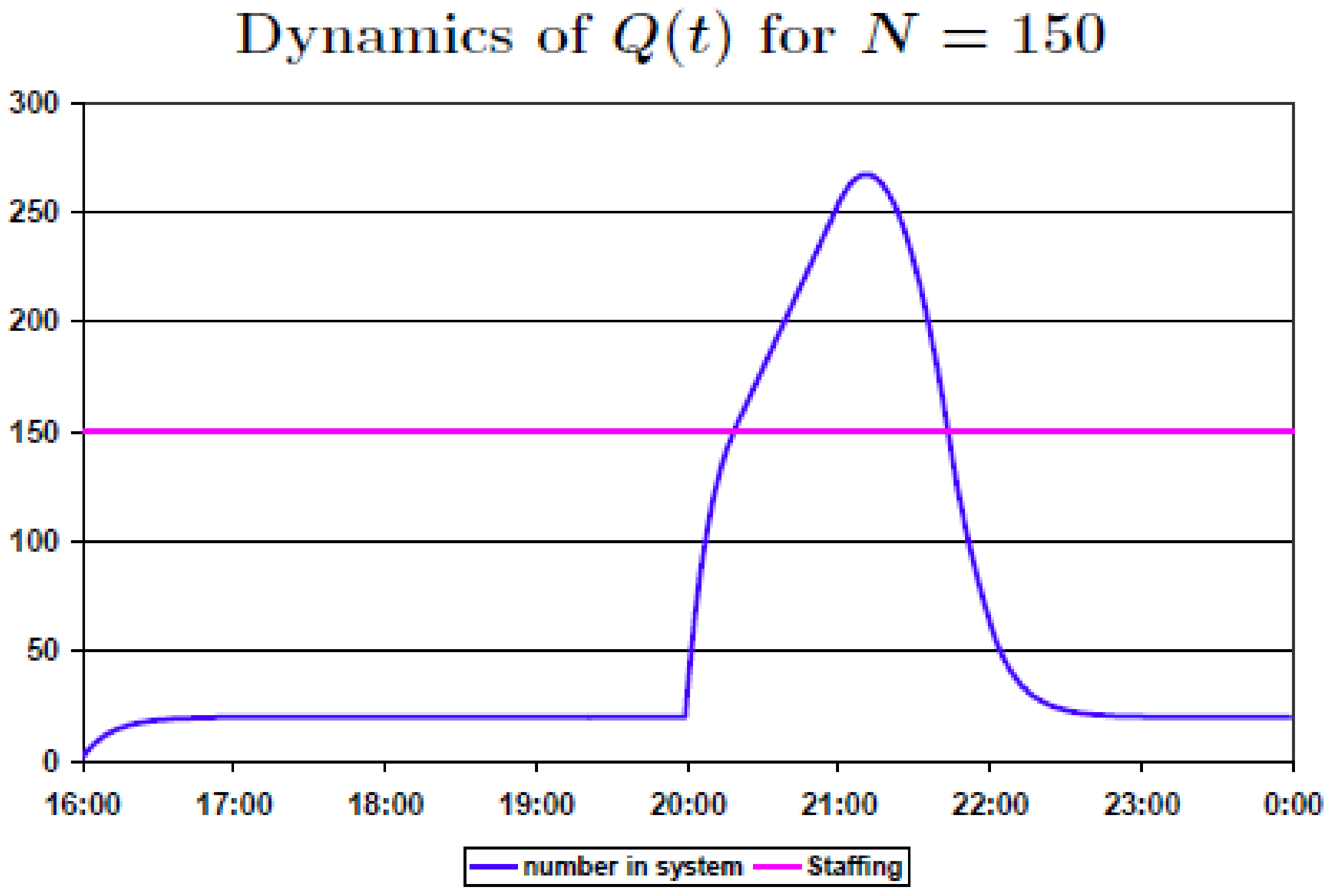
=$I$2*H13

=F12+C12-$D$2*MIN(F12,E12)

=MAX(F13-E13,0)

EX. Q(t) (2)

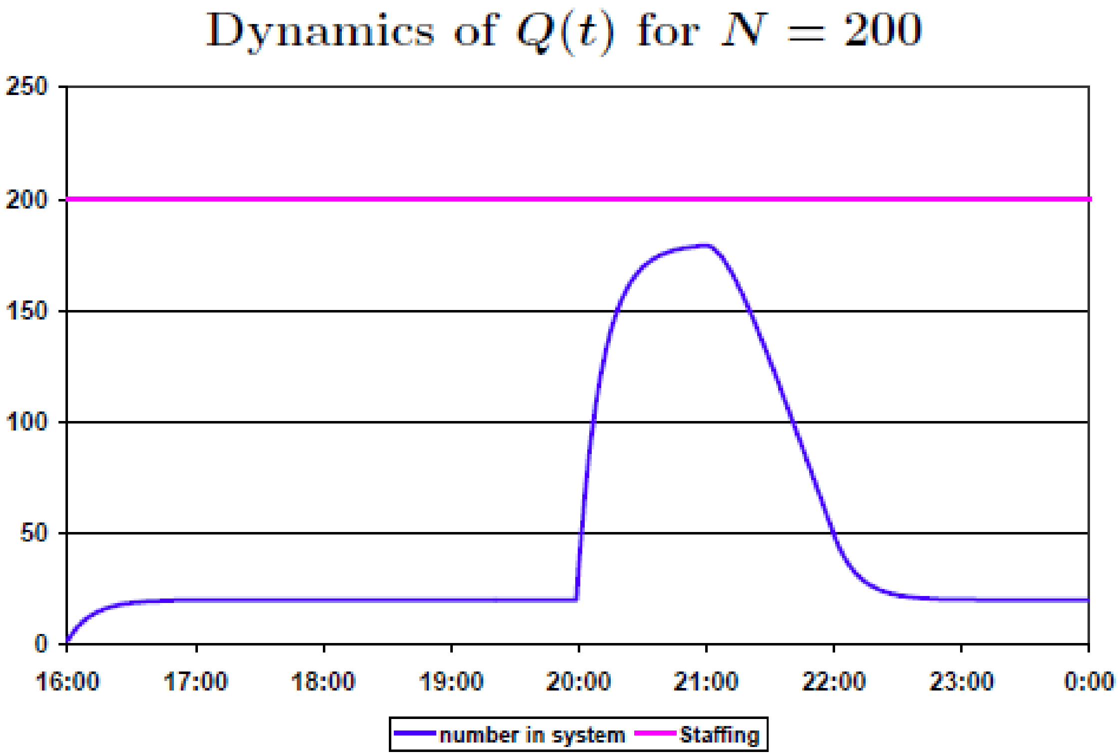
- N=150



Dynamics of $Q(t)$ for $N = 150$
300
250
200
150
100
50
0
16:00
17:00
18:00
19:00
20:00
21:00
22:00
23:00
0:00
number in system
Staffing

EX. Q(t) (3)

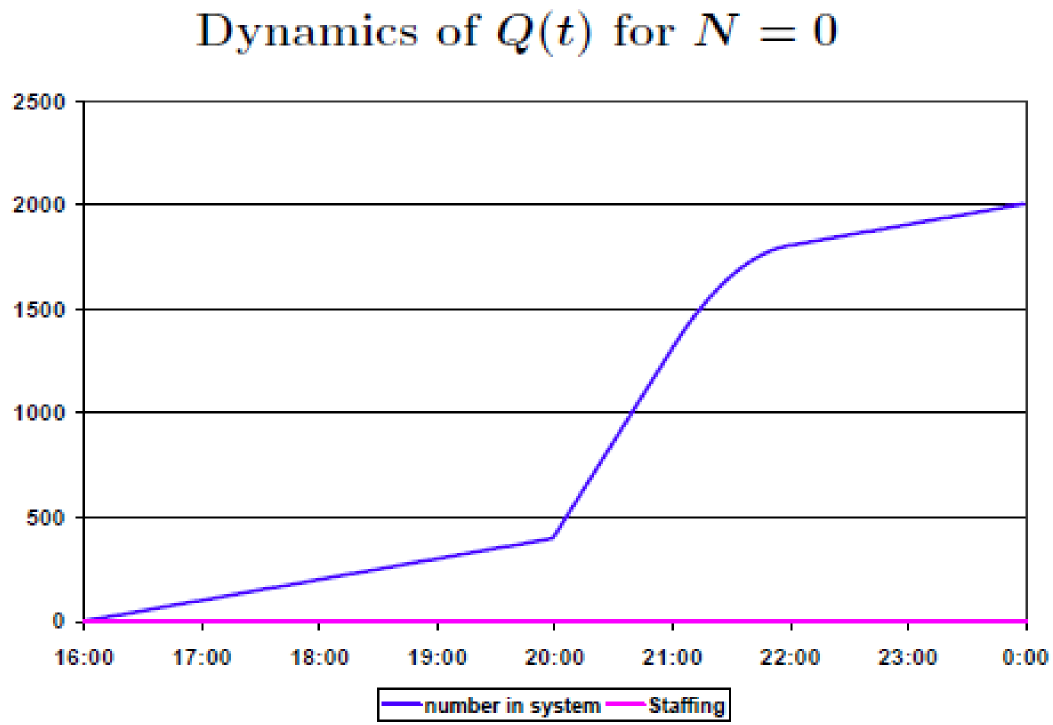
- N=200



- Q satisfies $\frac{d}{dt}Q(t) = \lambda(t) - \mu \cdot Q(t), \quad Q(0) = 0.$

EX. Q(t) (4)

- N=0

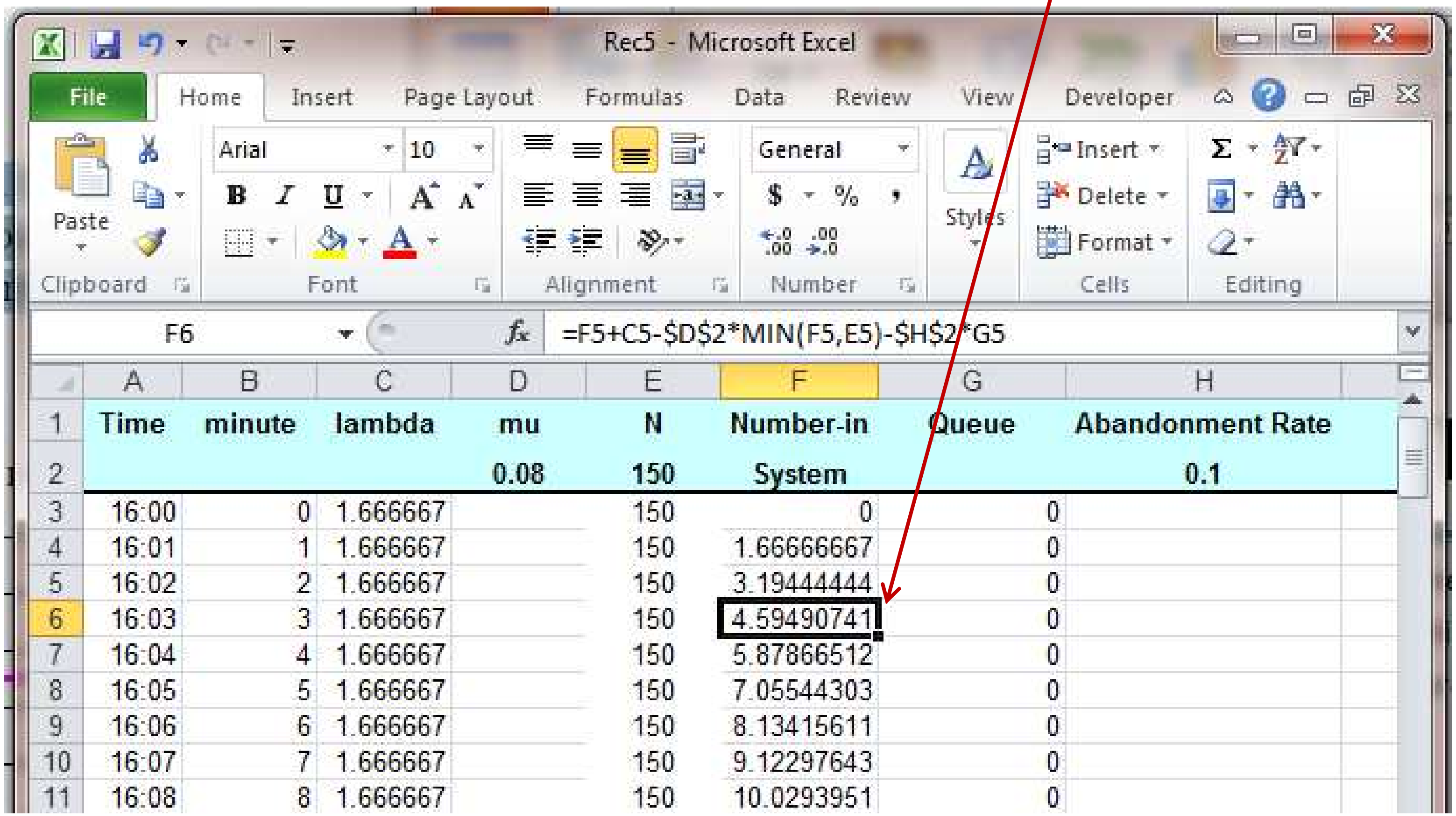


- $Q(t)$ satisfies $Q(t) = \int_0^t \lambda(t)dt \ (= A(t))$

EX. Q(t) with Abandonments

- Now assume that waiting customers can abandon from the system and the abandonment rate is $\theta =$ 6 customers per hour (i.e. each waiting customer abandons after an average of $\frac{1}{\theta}$ hours, if he was not admitted to service before).

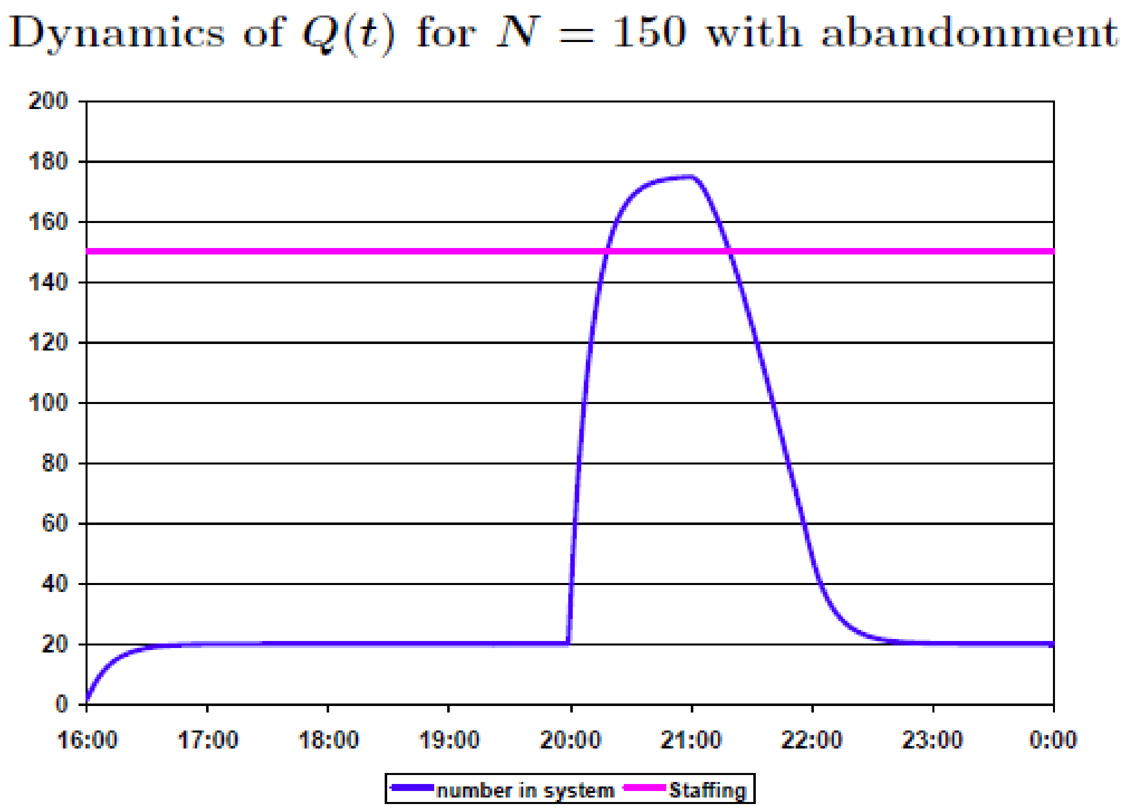
$$\boldsymbol{Q(6) = Q(5) + \lambda(5)\ \cdot \Delta t - \mu \cdot \big(Q(5) \wedge N(5)\big) \cdot \Delta t - \theta \cdot \big(Q(5) - N(5)\big)^{+} \cdot \Delta t}$$



	A	B	C	D	E	F	G	H
1	Time	minute	lambda	mu	N	Number-in	Queue	Abandonment Rate
2				0.08	150	System		0.1
3	16:00	0	1.666667		150	0	0	
4	16:01	1	1.666667		150	1.66666667	0	
5	16:02	2	1.666667		150	3.19444444	0	
6	16:03	3	1.666667		150	4.59490741	0	
7	16:04	4	1.666667		150	5.87866512	0	
8	16:05	5	1.666667		150	7.05544303	0	
9	16:06	6	1.666667		150	8.13415611	0	
10	16:07	7	1.666667		150	9.12297643	0	
11	16:08	8	1.666667		150	10.0293951	0	

EX. Q(t) with Abandonments (2)

- N=150



$$\frac{d}{dt}Q(t) = \lambda(t) - \mu \cdot \big(Q(t) \wedge N(t)\big) - \theta \cdot \big(Q(t) - N(t)\big)^{+}$$

EX. Profit Maximization

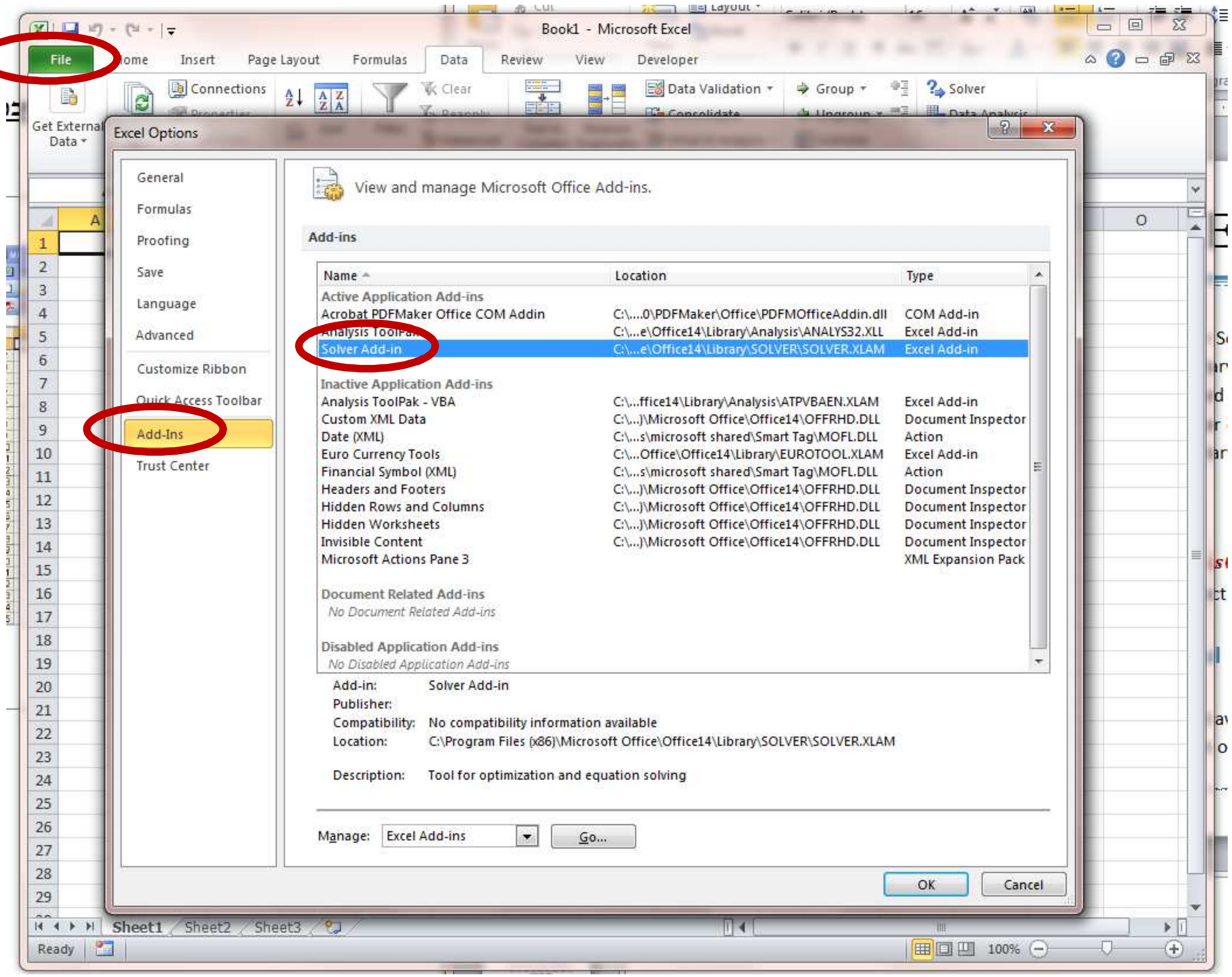
- Define
  - c - staffing cost rate (monetary unit per one unit of work)
  - r - service completion reward per customer
  - s - abandonment penalty per customer
  - h - waiting cost rate (monetary unit per unit of waiting time)

- Then the total profit is
  - $\boldsymbol{C^{(N)} = \int_0^T [r\mu(Q_t \wedge N_t) - (s\theta + h)(Q_t - N_t)^+ - cN_t]dt}$
  - Depends on the staffing function $N = \{N(t), 0 \leq t \leq T\}$

- How to choose the **optimal staffing** N=N(t) in order to maximize the profit $C^{(N)}$?
  - Exact solution is usually not available.
  - **EXCEL Solver** can be used in order to compute an approximate solution.

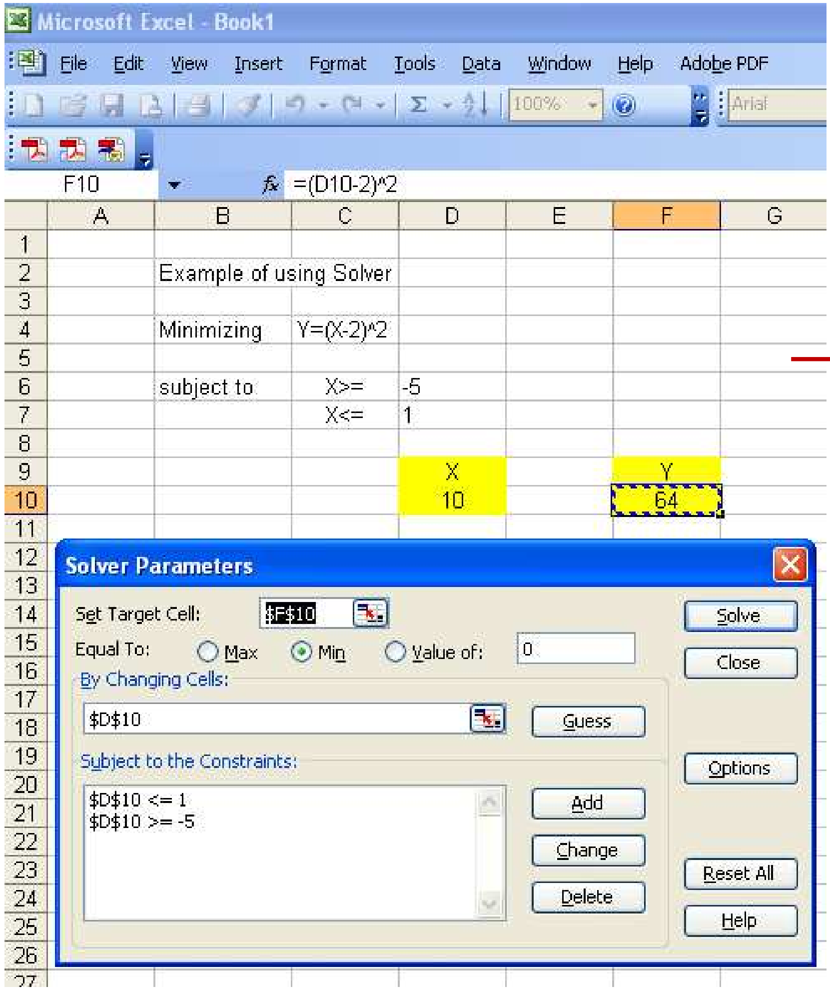
EX. EXCEL Solver

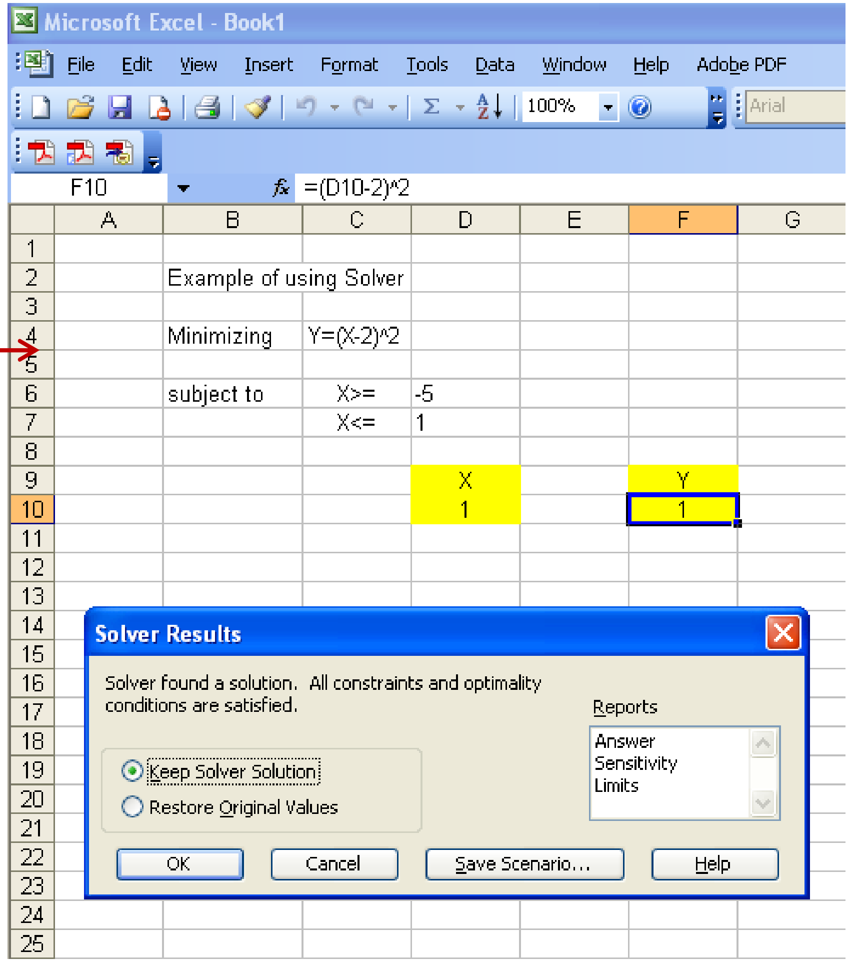
- Activate **Solver** Add-in
  - File -> Options -> Add-Ins -> Solver Add-in



EX. EXCEL Solver (2)

- Illustrative example





EX. Profit Maximization (2)

- Case 1: Assume N fixed. Minimize the cost of running the call center
  - An hour work of a server costs \$48 -> c =0.8
  - Minute waiting of a customer costs \$1 -> h=1
  - Assume no abandonments (θ=0) and no rewards (r=0)

  - Use Solver to solve this problem

=SUM(G4:G483)+SUM(I4:I483)

	A	B	C	D	E	F	G	H	I	J
1	Time	minute	lambda	mu	N	Number-in	Staffing Cost	Queue	Waiting Cost	Cost
2				0.08	50	System	0.8		1	174343.43
3	16:00	0	1.666667		50	0.000		0		
4	16:01	1	1.666667		50	1.667	40	0	0	
5	16:02	2	1.666667		50	3.194	40	0	0	
6	16:03	3	1.666667		50	4.595	40	0	0	
7	16:04	4	1.666667		50	5.879	40	0	0	
8	16:05	5	1.666667		50	7.055	40	0	0	
9	16:06	6	1.666667		50	8.134	40	0	0	
10	16:07	7	1.666667		50	9.123	40	0	0	
11	16:08	8	1.666667		50	10.029	40	0	0	
12	16:09	9	1.666667		50	10.860	40	0	0	
13	16:10	10	1.666667		50	11.622	40	0	0	

=100/6

=\$E\$2

=\$G\$2*E13

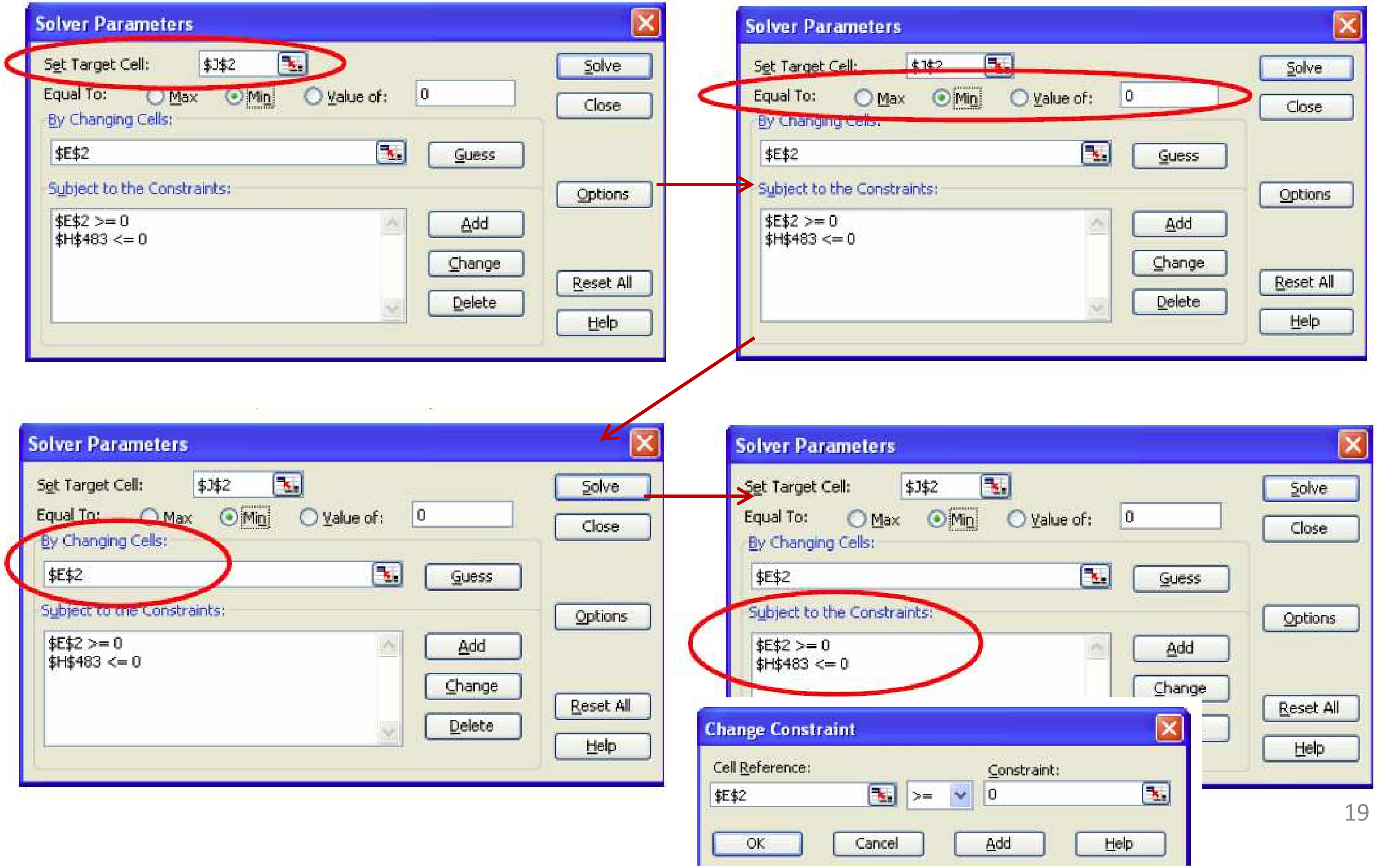
=\$I\$2*H13

=F12+C12-\$D\$2*MIN(F12,E12)

=MAX(F13-E13,0)

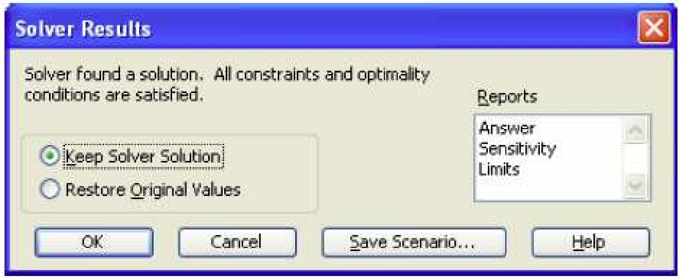
EX. Profit Maximization (3)

- Case 1 continued



EX. Profit Maximization (4)

- Case 1 continued

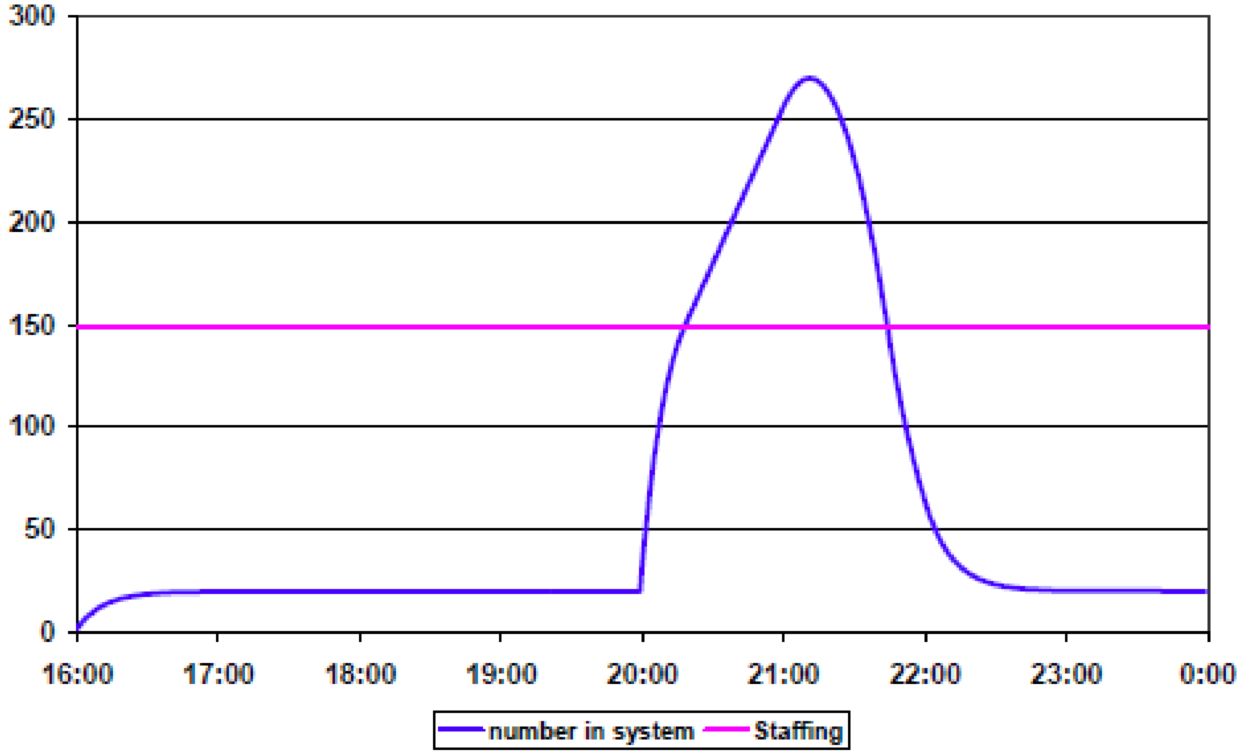


	C	D	E	F	G	H	I	J
1	lambda	mu	N	Number-in	Staffing Cost	Queue	Waiting Cost	Cost
2		0.08	149.3707	System	0.8		1	63534.19
3	1.666667		149.3707	0.000		0		
4	1.666667		149.3707	1.667	119.4965457	0	0	
5	1.666667		149.3707	3.194	119.4965457	0	0	
6	1.666667		149.3707	4.595	119.4965457	0	0	
7	1.666667		149.3707	5.879	119.4965457	0	0	

EX. Profit Maximization (5)

- Case 1 continued

Solution: The optimal staffing $N^* = 149$ and the cost $C^{N^*} = 63,534$.



EX. Profit Maximization (6)

- Case 2: Assume that the staffing can be changed at each hour.

	A	B	C	D	E	F	G	H	I	J	K	L
1	Time	minute	lambda	mu	N	Number-in	Staffing Cost	Queue	Waiting Cost	Staffing	Hourly	Cost
2				0.08		System	0.8		1	Time	Staffing	1[illegible]4473.43
3	16:00	0	1.666667		50	0		0		16:00-17:00	50	
4	16:01	1	1.666667		50	1.66666667	40	0	0	17:00-18:00	20	
5	16:02	2	1.666667		50	3.19444444	40	0	0	18:00-19:00	50	
6	16:03	3	1.666667		50	4.59490741	40	0	0	19:00-20:00	50	
7	16:04	4	1.666667		50	5.87866512	40	0	0	20:00-21:00	50	
8	16:05	5	1.666667		50	7.05544303	40	0	0	21:00-22:00	40	
9	16:06	6	1.666667		50	8.13415611	40	0	0	22:00-23:00	30	
10	16:07	7	1.666667		50	9.12297643	40	0	0	23:00-24:00	20	
11	16:08	8	1.666667		50	10.0293951	40	0	0			

k3

k4

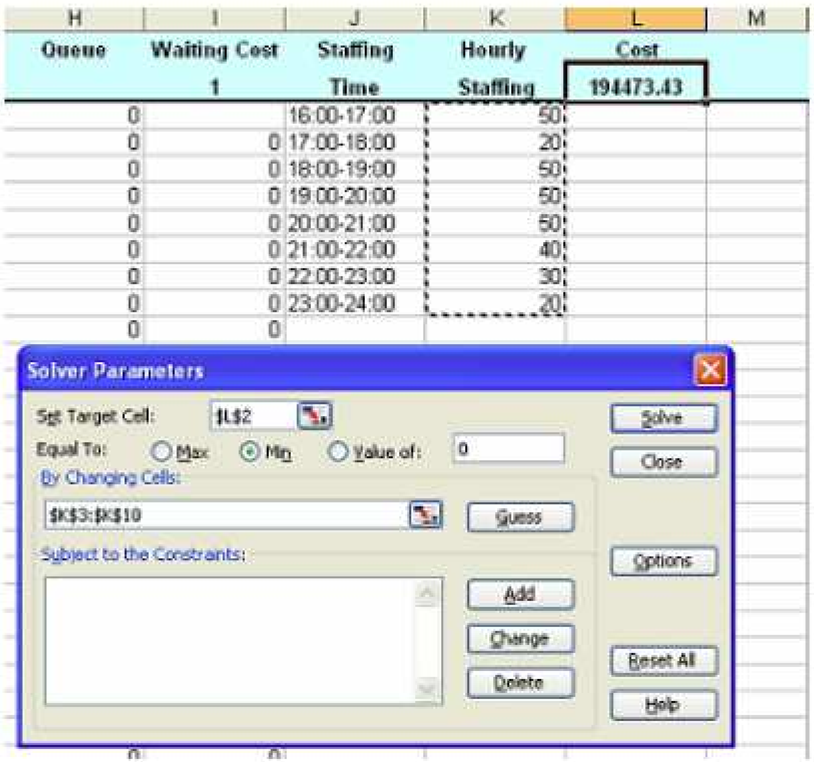
=k3

	A	B	C	D	E	F	G	H	I	J	K	L
1	Time	minute	lambda	mu	N	Number-in	Staffing Cost	Queue	Waiting Cost	Staffing	Hourly	Cost
2				0.08		System	0.8		1	Time	Staffing	194473.43
60	16:57	57	1.666667		50	19.860	40	0	0			
61	16:58	58	1.666667		50	19.871	40	0	0			
62	16:59	59	1.666667		50	19.882	40	0	0			
63	17:00	60	1.666667		50	19.892	40	0	0			
64	17:01	61	1.666667		20	19.901	16	0	0			
65	17:02	62	1.666667		20	19.909	16	0	0			
66	17:03	63	1.666667		20	19.917	16	0	0			

=k4

EX. Profit Maximization (7)

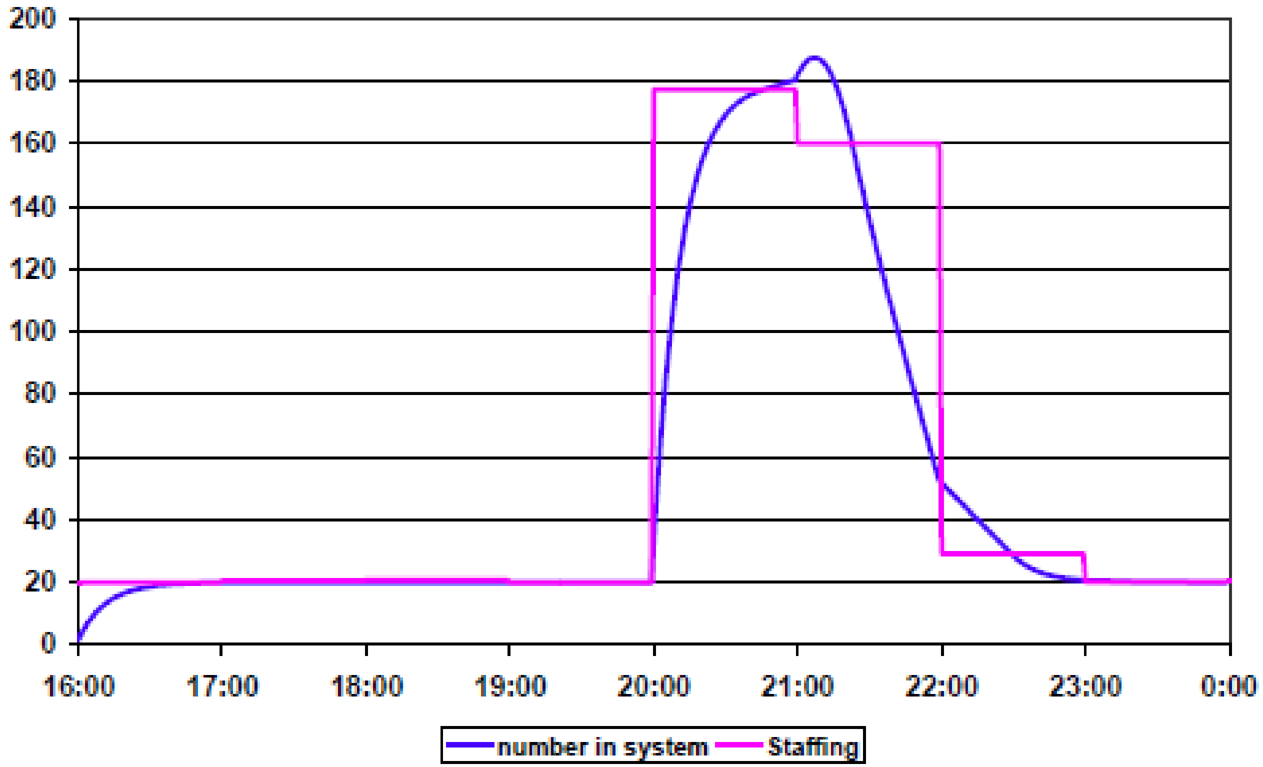
- Case 2 continued



EX. Profit Maximization (8)

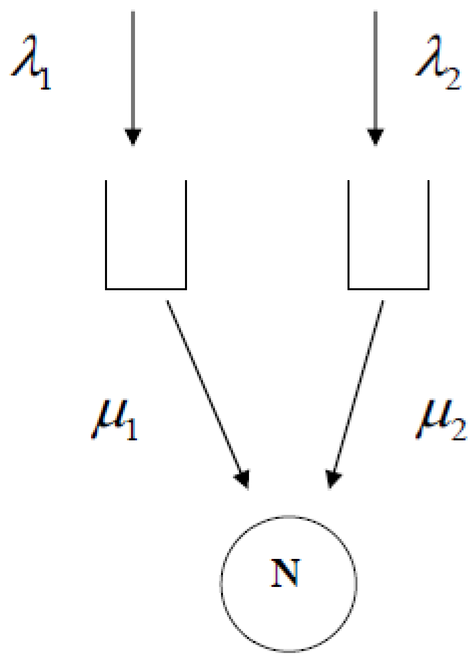
- Case 2 continued

Solution : The optimal cost $C^{N^*} = 23,133.$



Part 3. V-model

- Assume there are two classes of customers
  - VIP (Class 1), with arrival rate $\lambda_1(t)$
  - Regular (Class 2), with arrival rate $\lambda_2(t)$
  - Each server serves both classes, with rates μ_i for class i (i=1,2)
  - Define $Q_i(t)$ to be the total number of class i customers in the system, i=1,2



- Let $Q(t) = Q_1(t) + Q_2(t)$. Assume $Q(0) = 0$.
  - Is it possible to write the differential equation for Q(t) without any additional assumptions?

Part 3. V-model

- A routing policy: assume that the call center works in the **preemptive-resume regime**:
  - At every moment a service to a customer can be interrupted (in this case a customer goes back to queue of its class) and resumed at a later time.
  - VIP customers are *high priority customers*, which means that no regular customer can be in service while VIP customer is waiting.

- Differential equation for $Q_i(t), i = 1, 2$
  - $\frac{d}{dt} Q_1(t) = \lambda_1(t) - \mu_1 \cdot (Q_1(t) \wedge N(t)),\ Q_1(0) = 0,$
  - $\frac{d}{dt} Q_2(t) = \lambda_2(t) - \mu_2 \cdot \left(Q_2(t) \wedge \left(N(t) - Q_1(t)\right)^+\right),\ Q_2(0) = 0.$