Course Review: Introductory Part.

- Introduction to Services and Queues (Service Nets = Queueing Nets)

  Our Service Economy.

  Tele-Services (Telephone, Internet, email, Fax, Chat).

  Queues in service systems are here to stay (at least for a while).

  Operational Queues: Perpetual, Predictable, Stochastic.

- Measurements: The First Prerequisite

  Transaction-based (time-based) measurements.

  Face-to-Face, Telephone, Transportation, Internet, Administrative Services.

  Scenario Analysis (vs. Simulation or Analytical Models): very typical or rare event.

- Models: The Second Prerequisite

  Empirical Models: data-based; simple yet possibly far-reaching.

  The Skeptic (Flanders).

  vs. The Believer/Practitioner (Larson, our class).

- The Fluid View; A Deterministic Service-Station

  Averaging over many (similar enough) scenarios.

  Capacity/Bottleneck analysis (via spreadsheet,LP).

  Utilization Profiles for resources.

  Inventory Buildup Diagrams (via "National Cranberry" HBS case).

- The Processing Network Paradigm

  TQM (80's), continued by BPR (90') = Business Process ReEngineering.

  Dynamic Stochastic Project/Processing Networks (DSP-nets = DS-PERTs).

  Applications: Arrest-to-Arraignment, Israeli Electric Company, Multi-Project Management;

  Y Operational Q's: scarce resources; synchronization/coordination gaps, design constraints.

  Q1: Can we do it? via Bottleneck Analysis $\leftrightarrow$ the fluid view.

  Q2: How long will it take? typically via stochastic networks.

  Q3: Can we do better? via parametric/sensitivity/what-if analysis.

  Q4: How much better? via optimality/approximation analysis.

- Towards modelling a Stochastic Service Station: the main building blocks

  Arrivals' epochs: Poisson = *the* model for completely random arrivals.

  Service durations: within the Phase-type framework.

  Customers' patience

Service Engineering

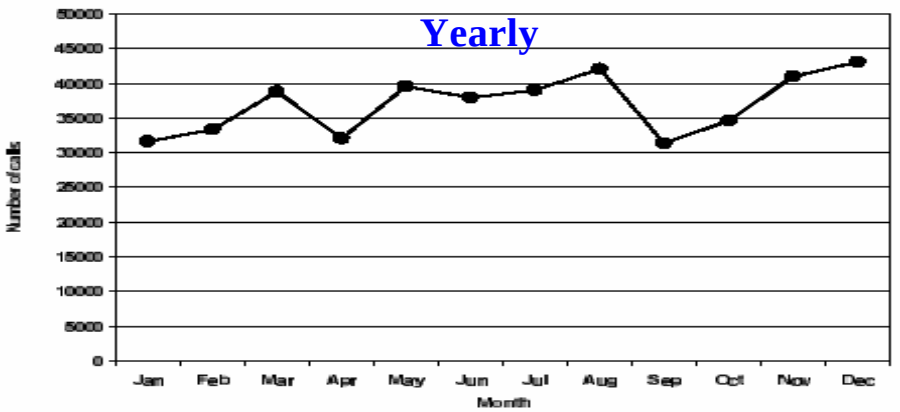
Class 6

Modeling Arrivals to a Service Station: The Poisson Process, and Relatives.

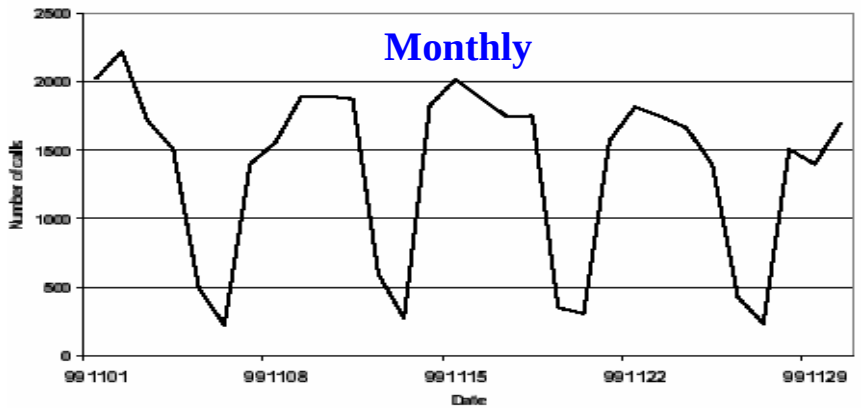
- Empirical Introduction, via DataMOCCA.
- The Poisson Process: 4 Definitions, Properties.
  - PASTA = Poisson Arrivals See Time Averages.
  - Biased Sampling.
- Animation: from Bernoulli to Poisson, or The Law of Rare Events.
- Non-homogeneous Poisson Processes.
- Testing: Poisson or not Poisson.
- Modeling Arrivals to a Service Station.
- Forecasting of the Arrival Rate.
- Poisson Alternatives: eg. Internet Applications (Heavy Tails, Long-Range Dependence).
- On Limits Theorems in Probability: SLLN, CLT, Rare Events.

Arrivals to a Call Center (Israel, 1999): Time Scales

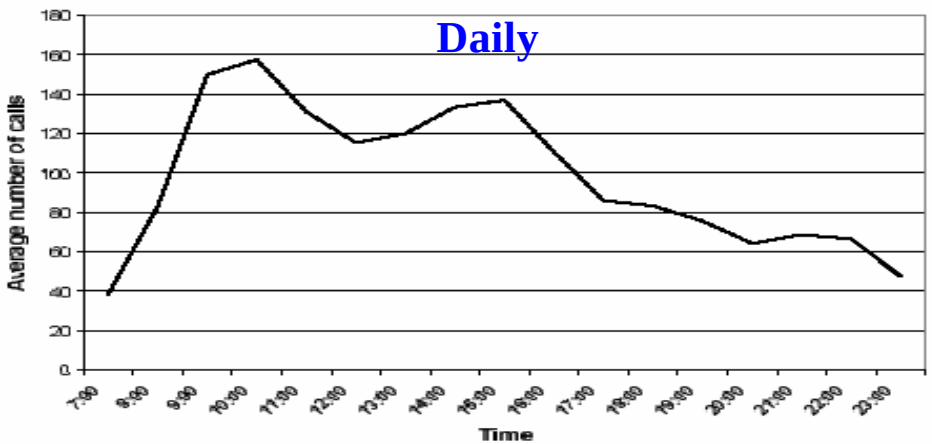
Strategic



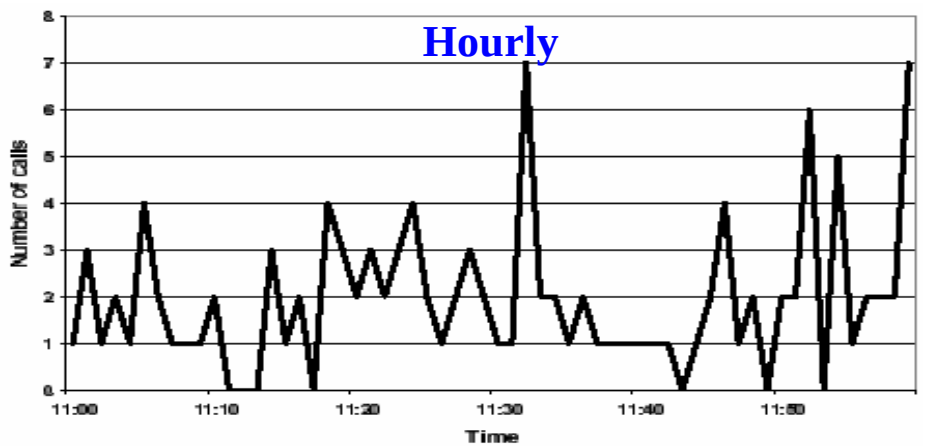
Tactical



Operational (Predictable Var.)



Regulatory (Stochastic)



Arrivals to a Call Center (U.S., 1976): Queueing Science

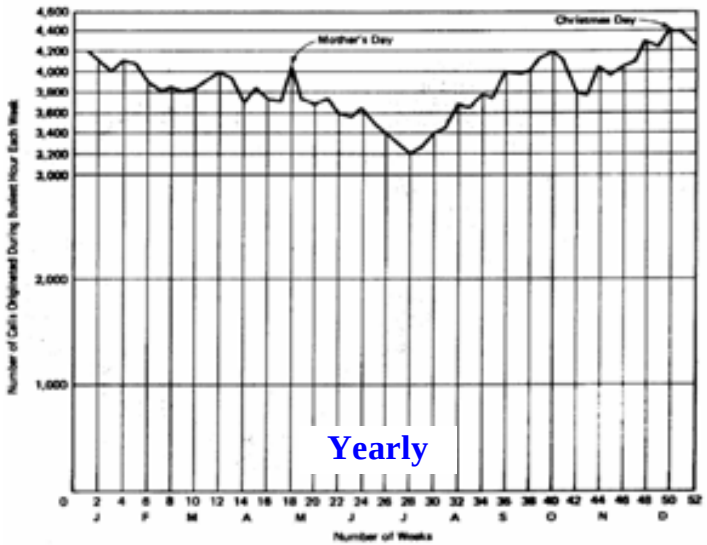


Figure 1 Typical distribution of calls during the busiest hour for each week during a year.

(E. S. Buffa, M. J. Cosgrove, and B. J. Luce, "An Integrated Work Shift Scheduling System")

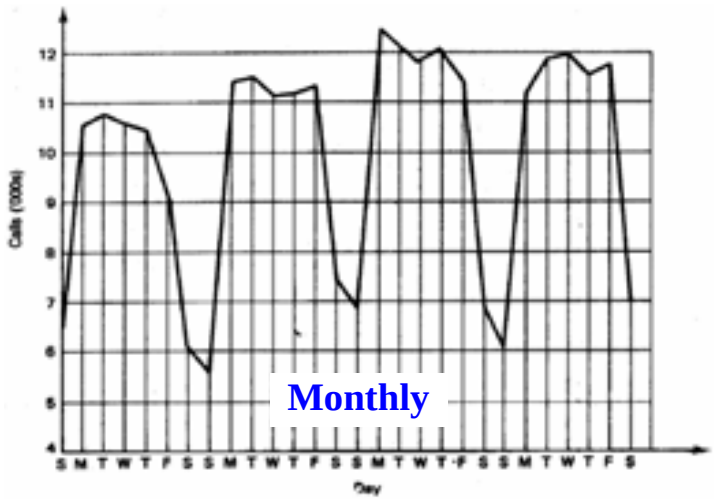


Figure 2 Daily call load for Long Beach, January 1972.

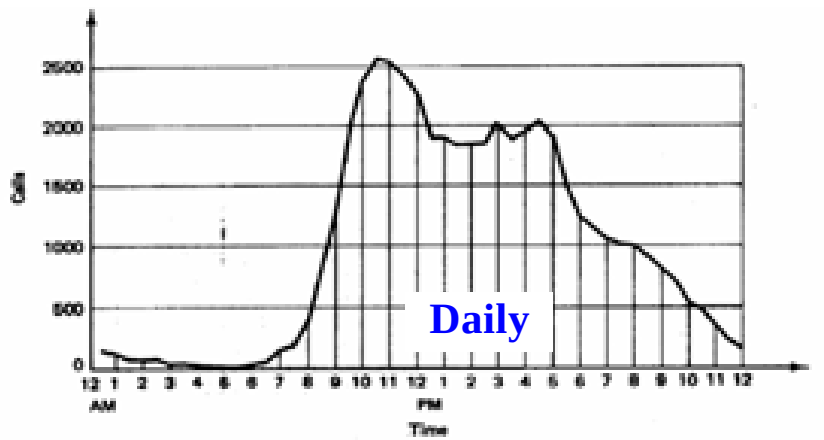


Figure 3 Typical half-hourly call distribution (Bundy D A).

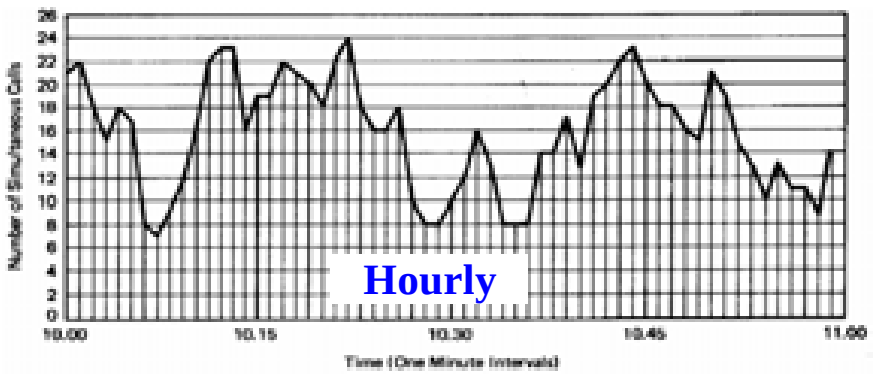
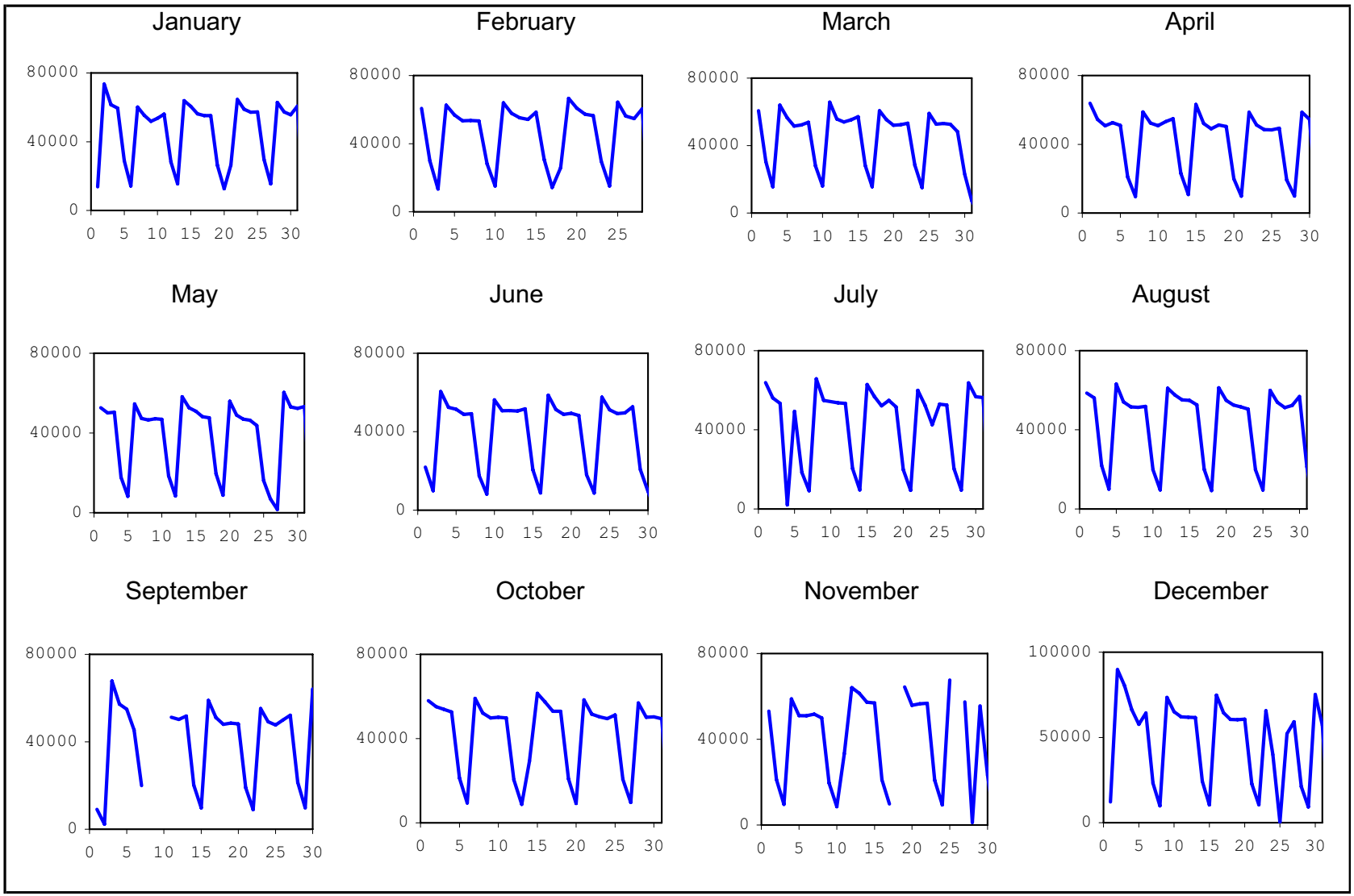


Figure 4 Typical intrahour distribution of calls, 10:00–11:00 A.M.

Monthly Arrivals to Service

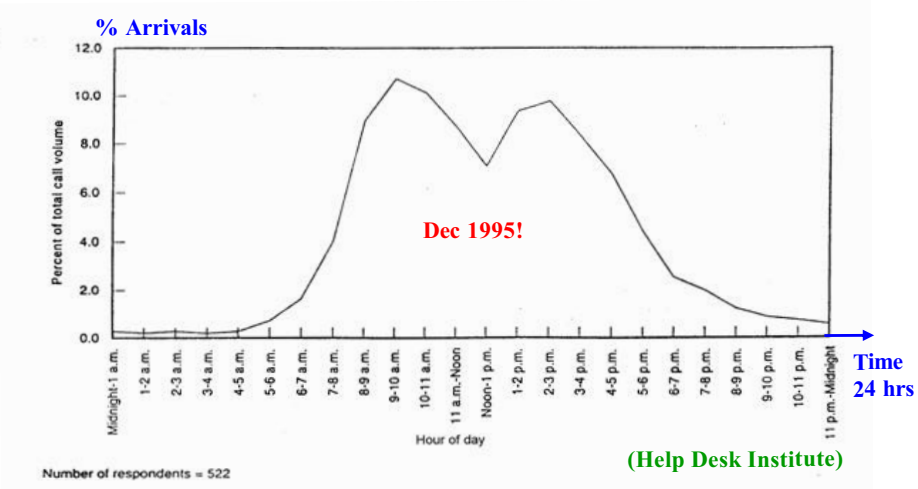
U.S. Bank: Daily Arrival-Rates, over a Month, 2002



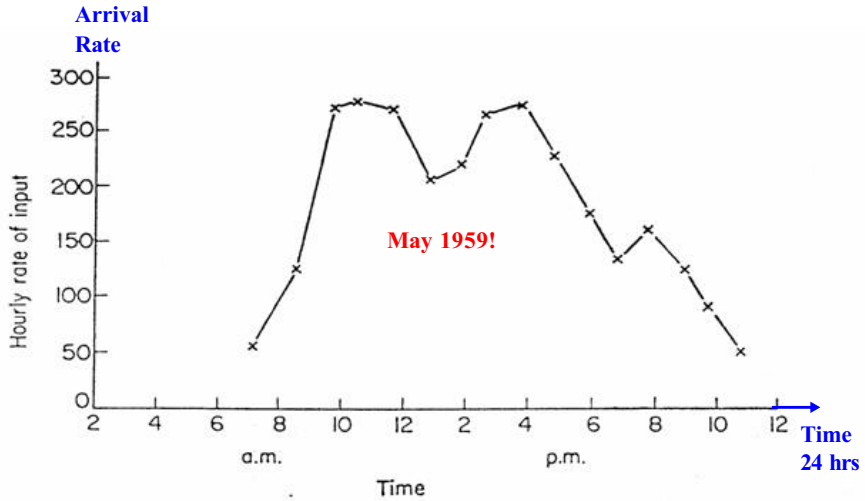
Daily Arrivals to Service: Time-Inhomogeneous (Poisson?)

Intraday Arrival-Rates (per hour) to Call Centers

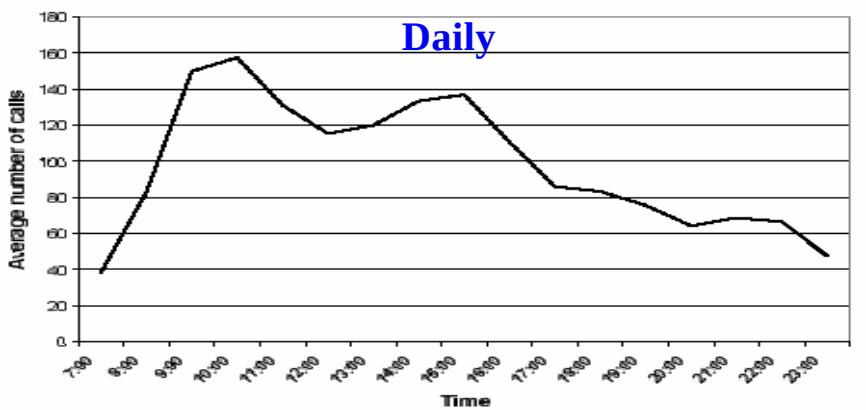
December 1995 (700 U.S. Helpdesks)



May 1959 (England)

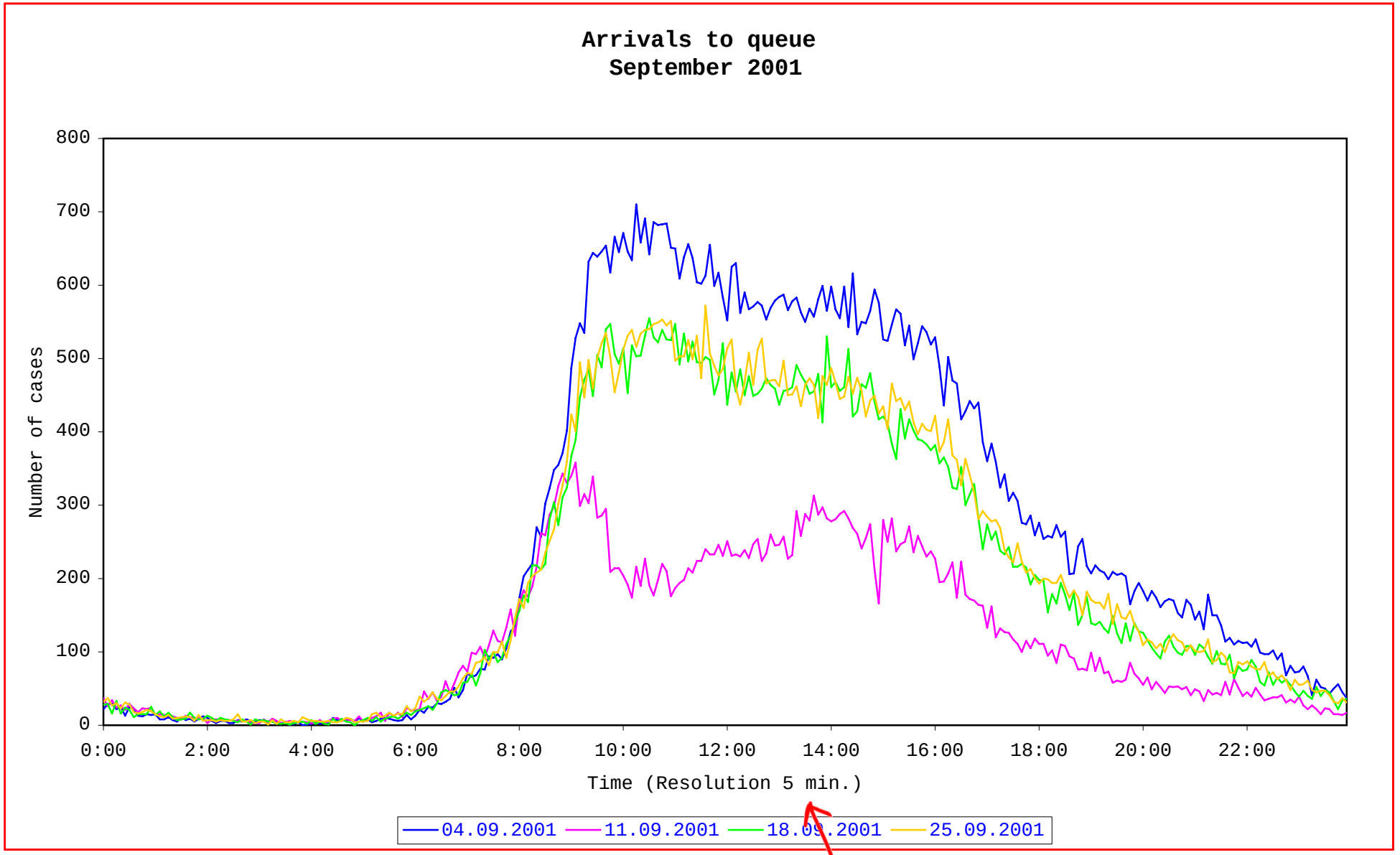


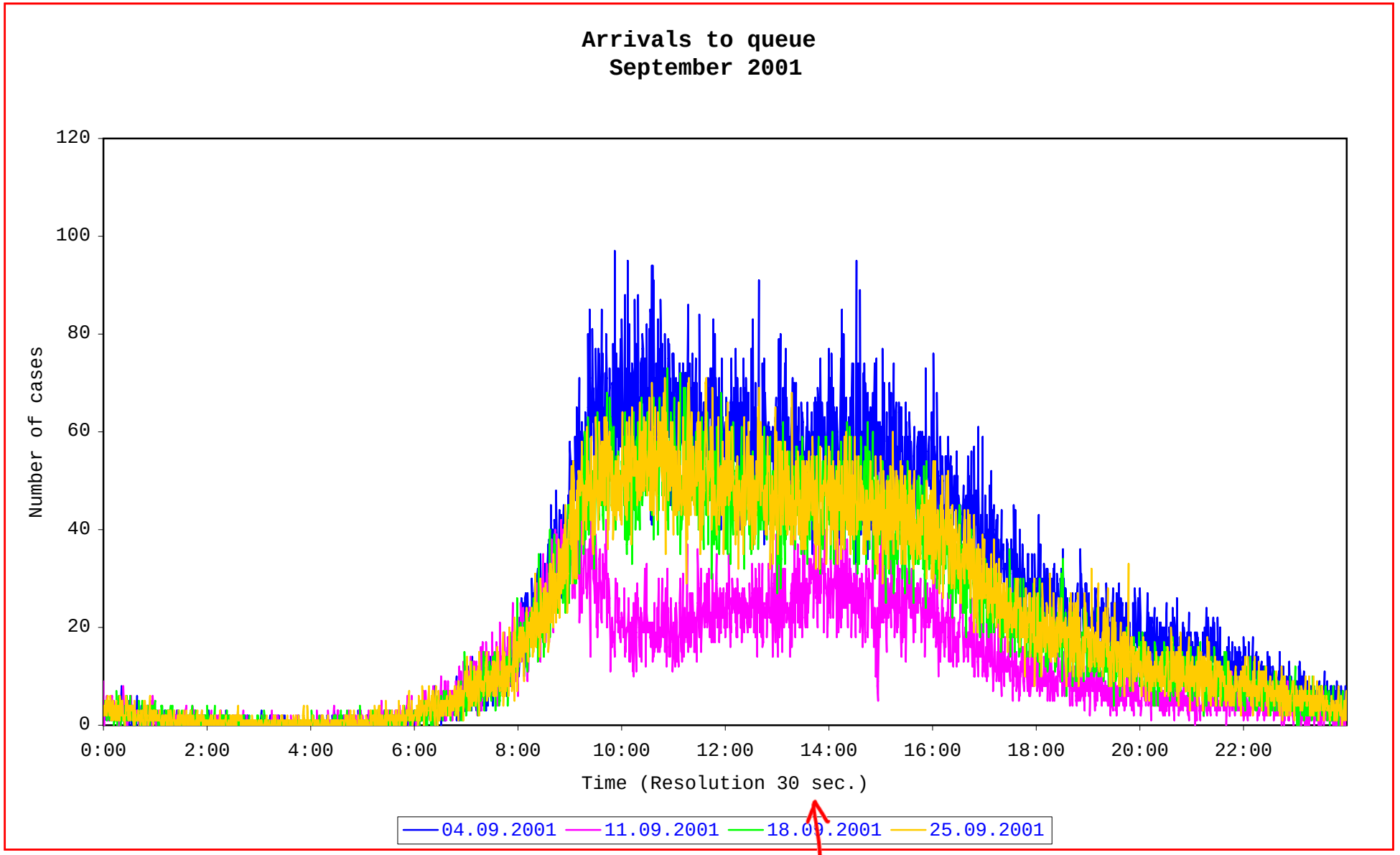
November 1999 (Israel)



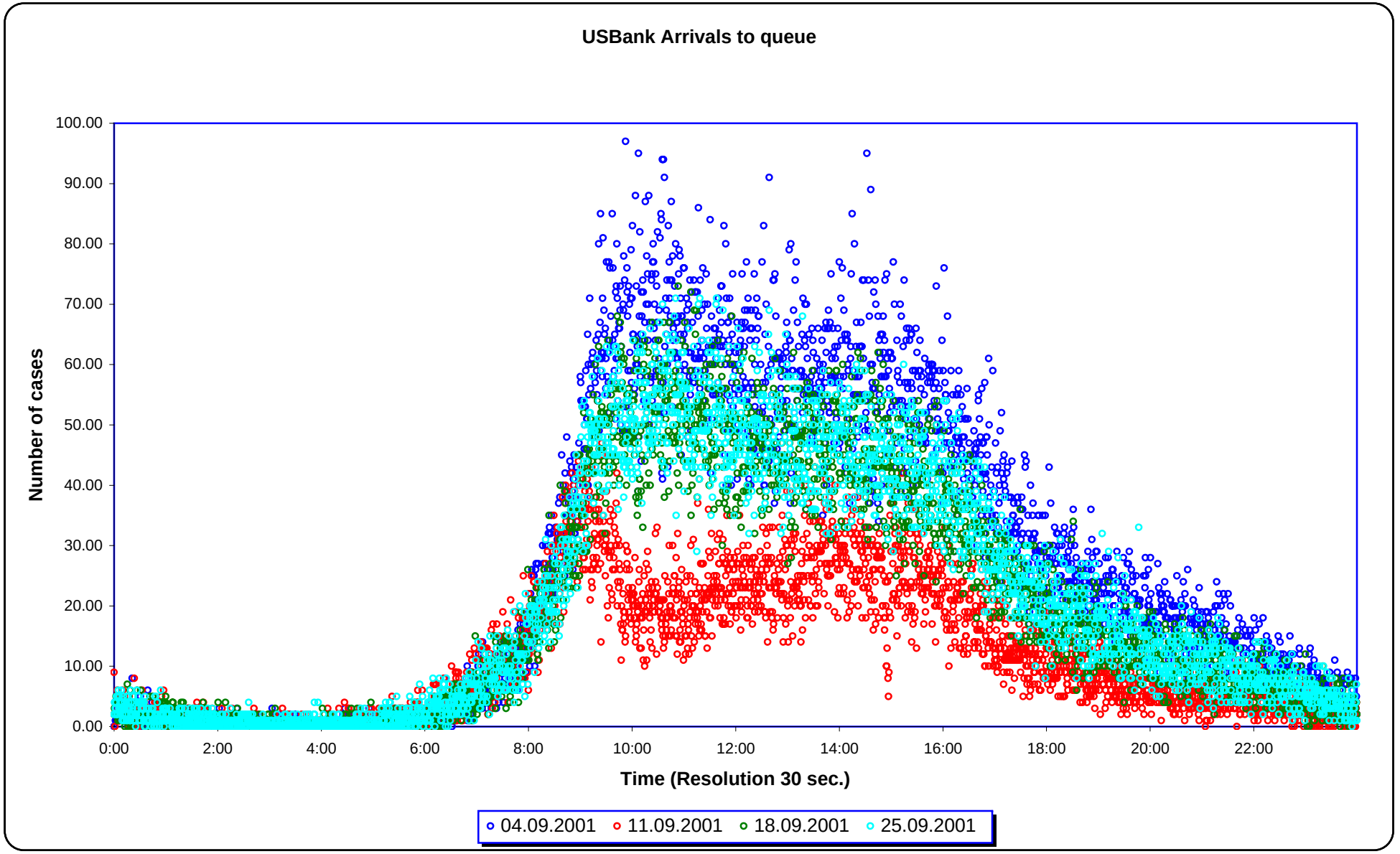
Observation:

Peak Loads at **10:00 & 15:00**


Arrivals to queue
September 2001
800
700
600
500
400
300
200
100
0
Number of cases
0:00
2:00
4:00
6:00
8:00
10:00
12:00
14:00
16:00
18:00
20:00
22:00
Time (Resolution 5 min.)
04.09.2001
11.09.2001
18.09.2001
25.09.2001
default

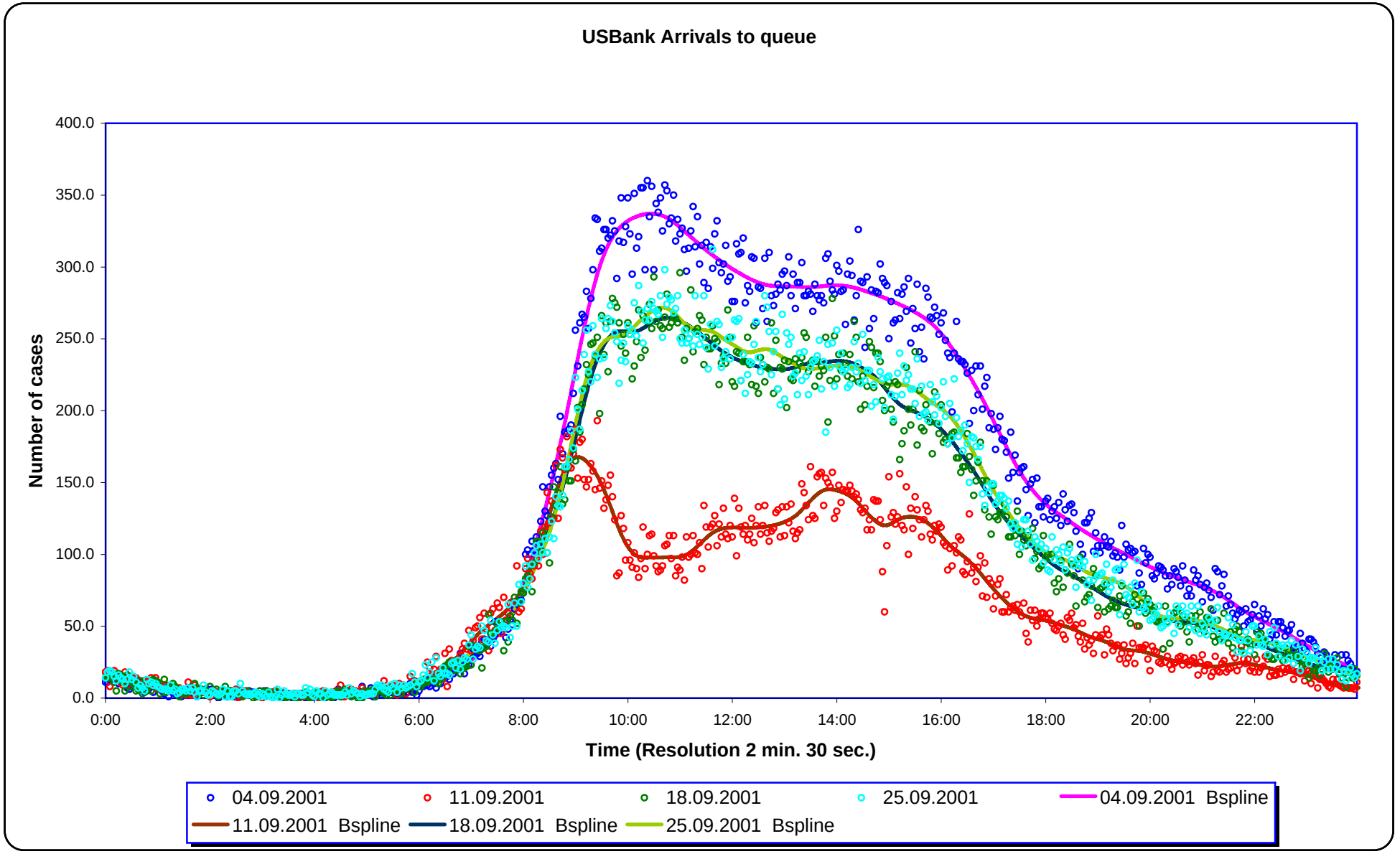

Arrivals to queue
September 2001
Number of cases
120
100
80
60
40
20
0
0:00
2:00
4:00
6:00
8:00
10:00
12:00
14:00
16:00
18:00
20:00
22:00
Time (Resolution 30 sec.)
04.09.2001
11.09.2001
18.09.2001
25.09.2001
Stochastic Variability

Scatter vs. Polygon (in SEE Stat)



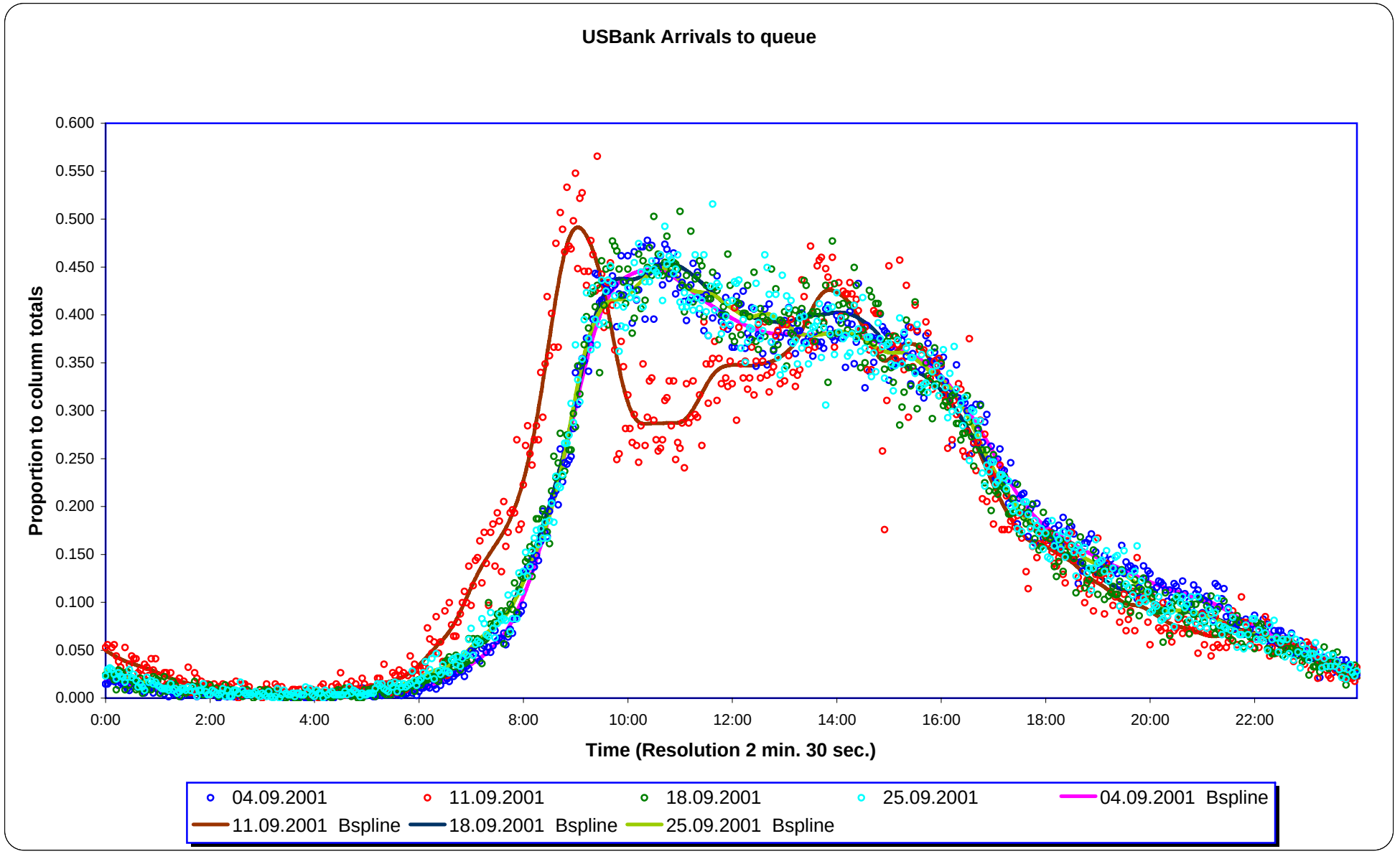
Observations : stochastic variability

Smoothing in SEEStat (Automatic)



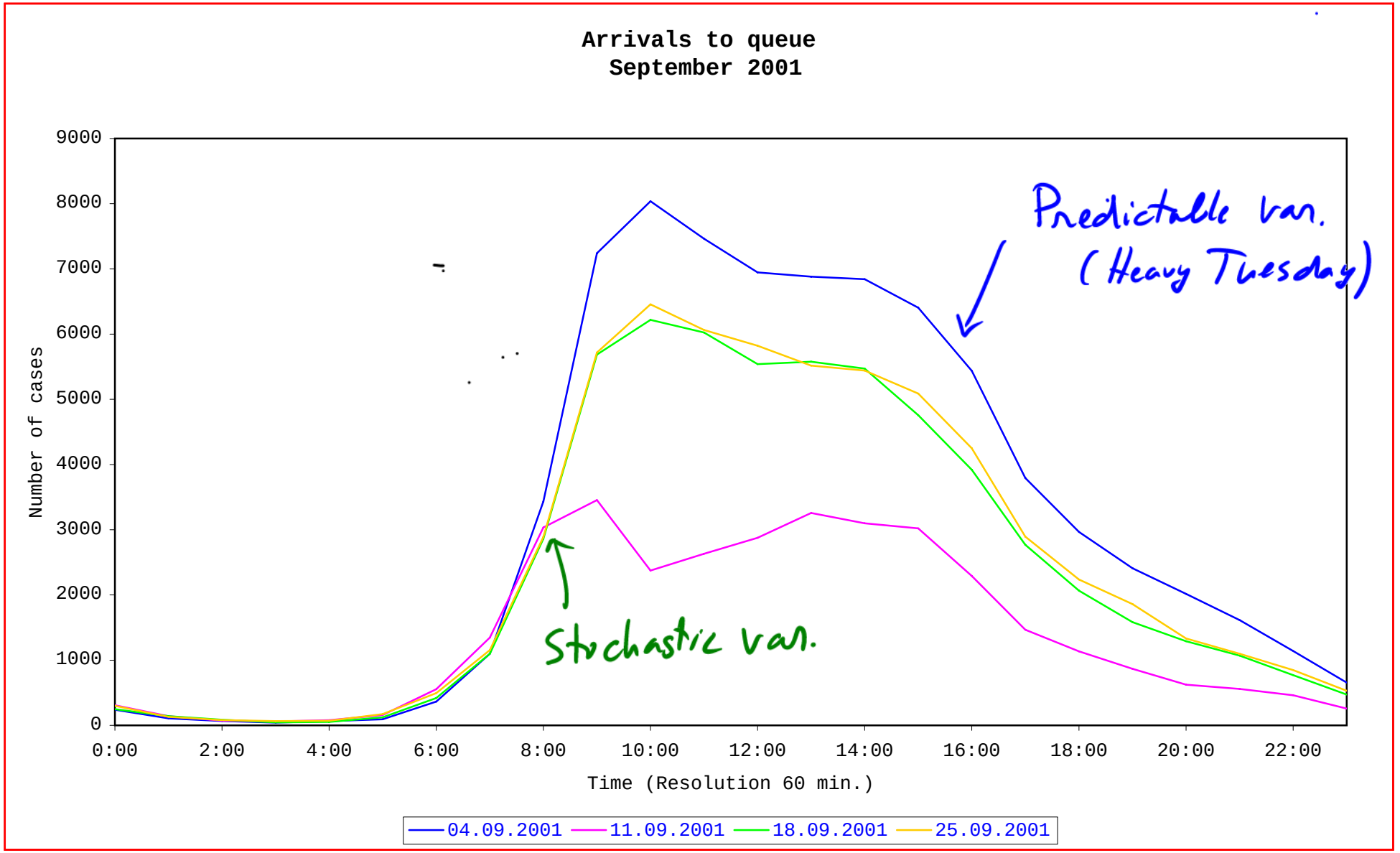
Smoothing : sophisticated averaging

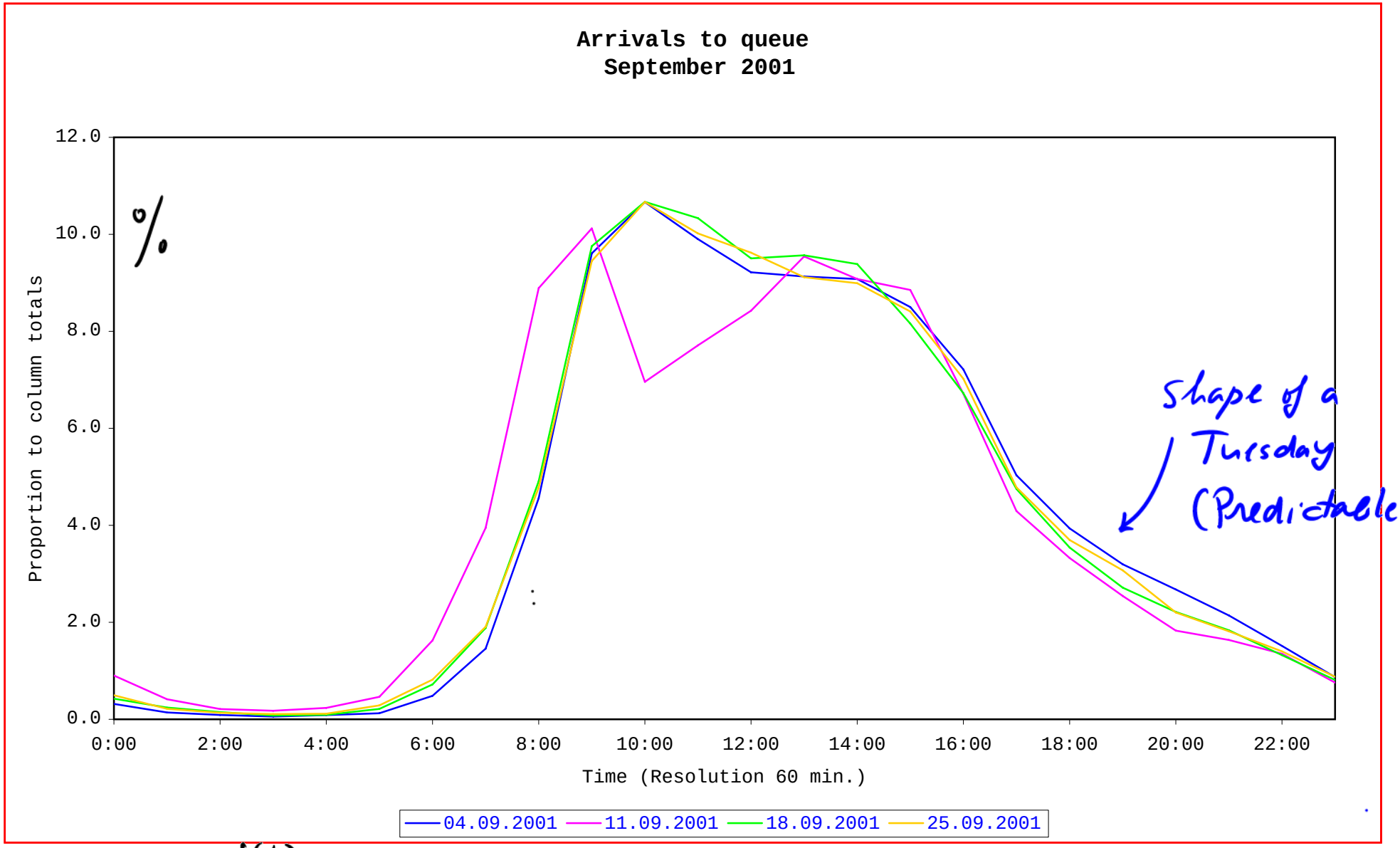
Predictable vs. Stochastic Variability



$\lambda(t) = \Lambda \times \lambda_0(t)$ $\Rightarrow$ Λ: Stochastic var. $\lambda_0(t)$: Predictable (shape)

30 sec → 1 hour : averaging

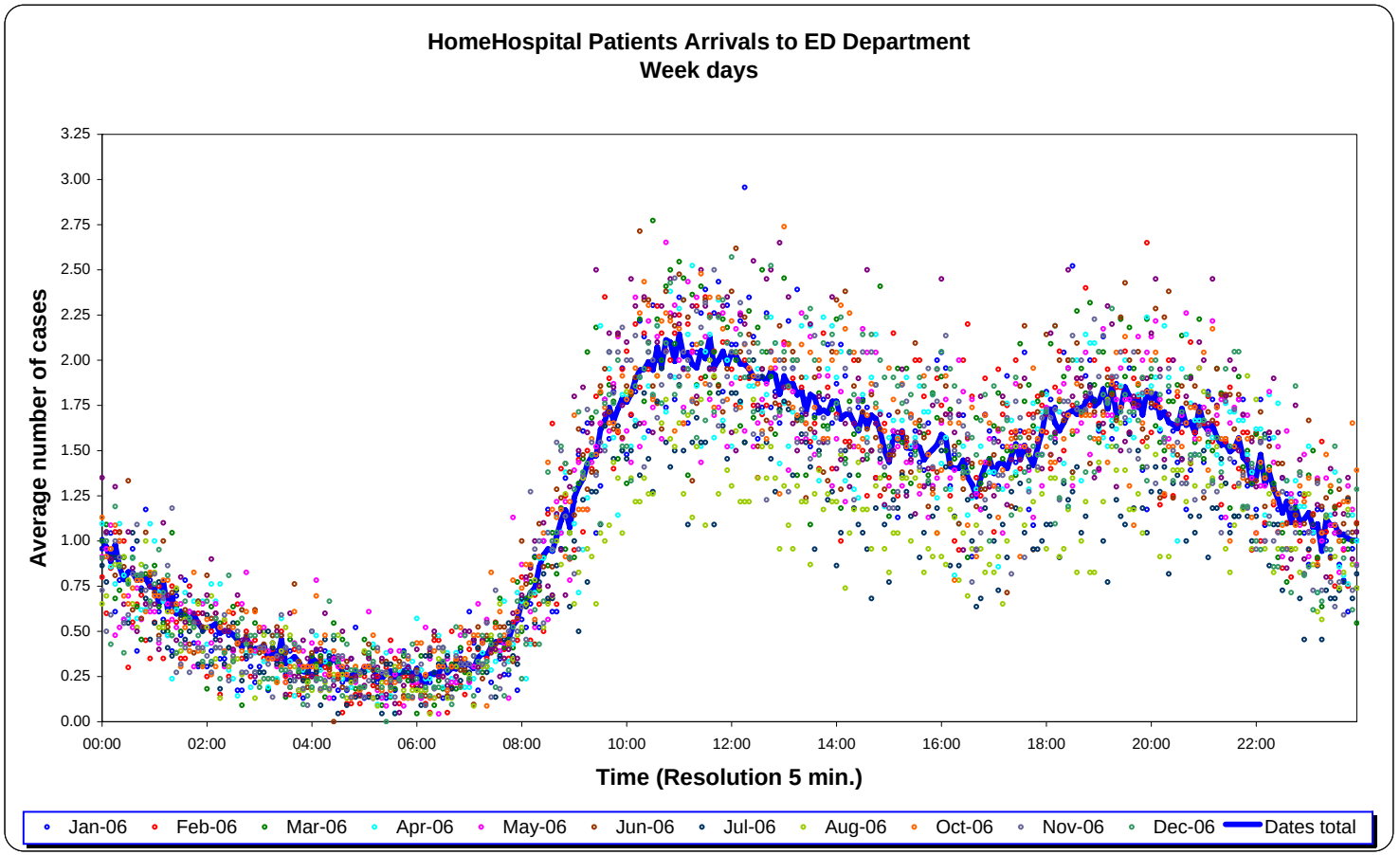




$$\lambda_{\%}(t) = \frac{\lambda(t)}{\int_0^T \lambda(u)\,du} = \frac{1}{T} \% \text{ of mean} \qquad \left(= \frac{\lambda(t)}{\sum_1^T \lambda(u)} \right)$$

Arrivals to an Emergency Department (ED)

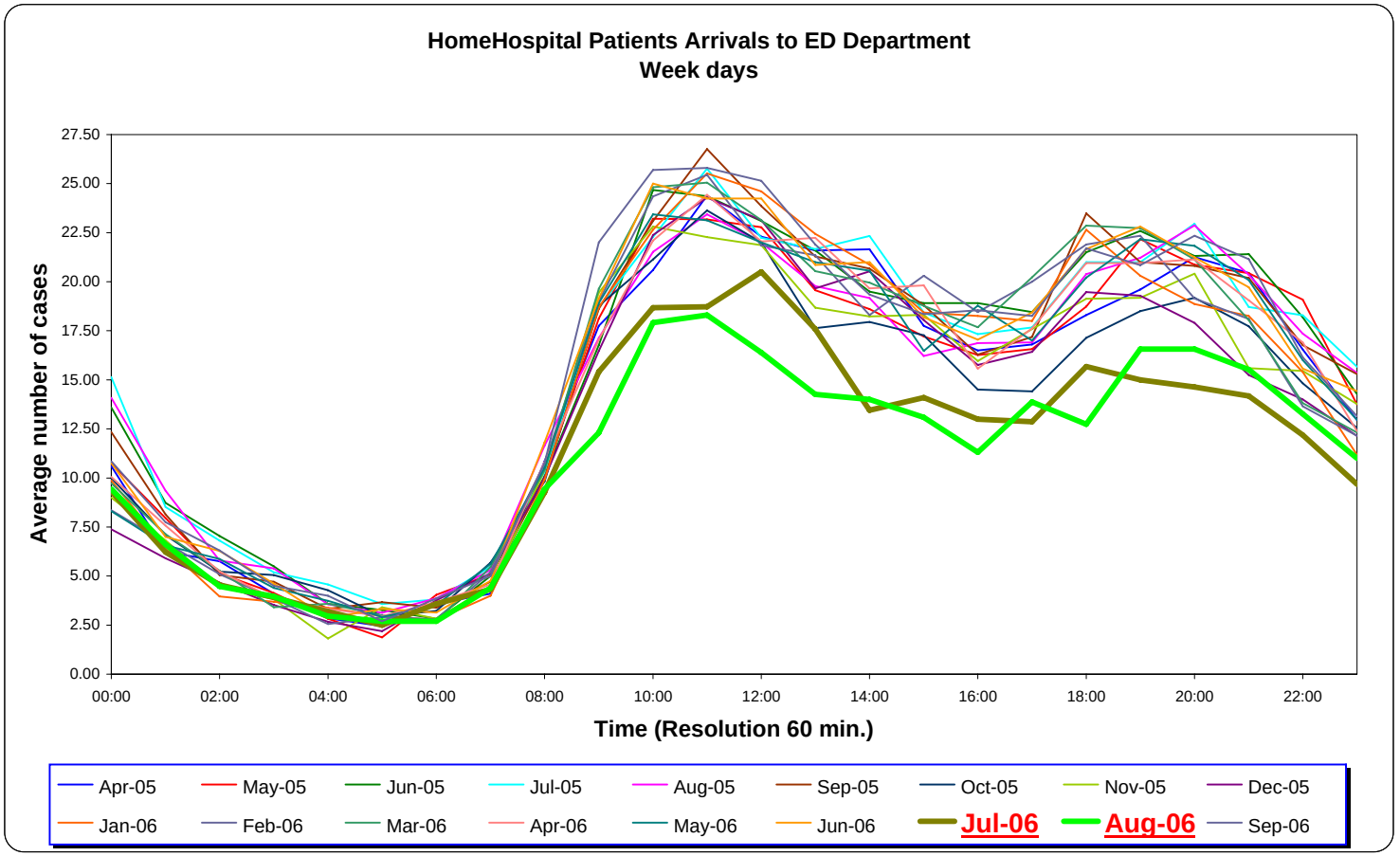
Large Israeli ED, 2006



- **Second** peak at 19:00 (vs. 15:00 in call centers).
- How much stochastic variability ?

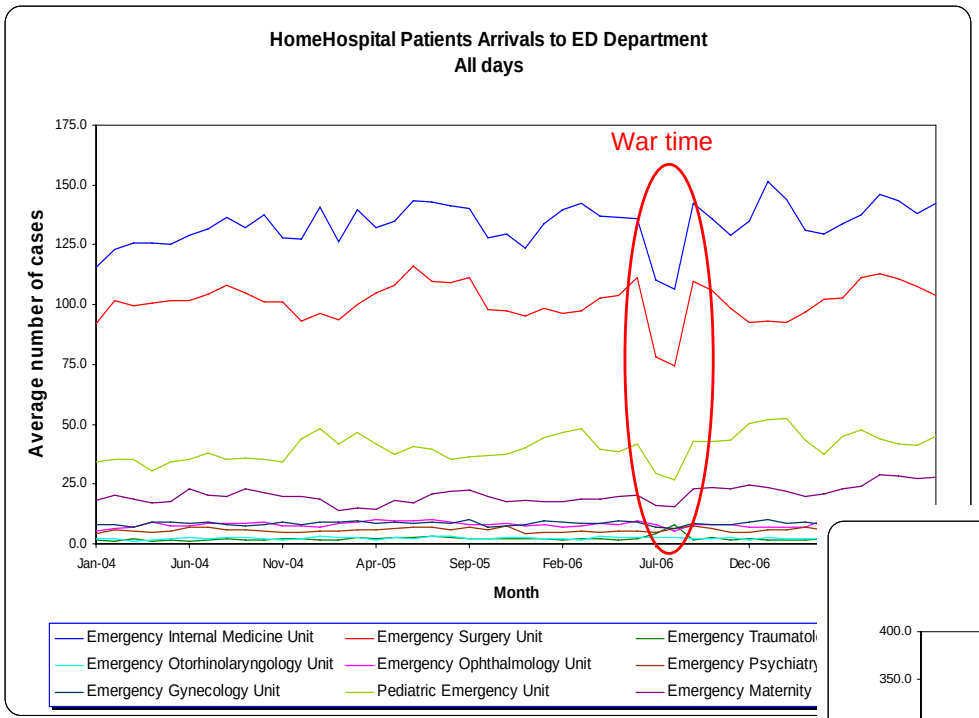
Arrivals to ED: Environment Dependence

Large Israeli ED, 2005-6

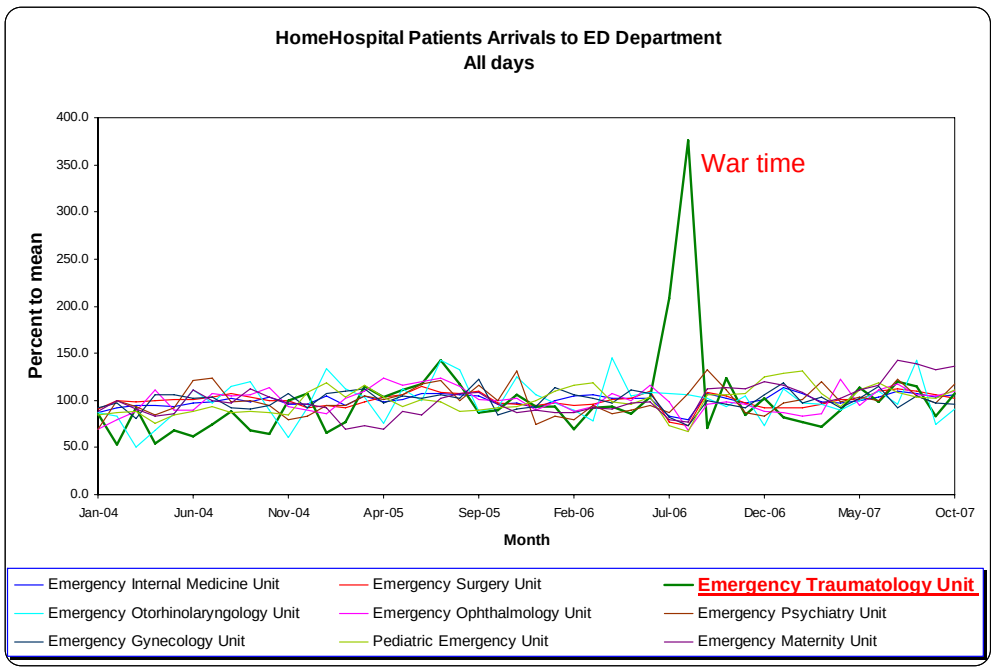


Arrivals to ED: Environment Dependence

Number of Arrivals



Percent to Mean



Predicting Emergency Department Status

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† *CSIRO Mathematical and Information Sciences, Private Bag 10, South Clayton MDC, Victoria 3169, Australia*

‡ *The Judge Institute of Management, University of Cambridge, Trumpington Street, Cambridge CB2 1AG, UK*

Abstract

Many acute hospitals in Australia experience frequent episodes of ambulance bypass. An important part of managing bypass is the ability to determine the likelihood of it occurring in the near future.

We describe the implementation of a computer program designed to forecast the likelihood of bypass. The forecasting system is designed to be used in an Emergency Department. In such an operational environment, the focus of the clinicians is on treating patients, there is no time carry out any analysis of the historical data to be used for forecasting, or to determine and apply an appropriate smoothing method.

The method is designed to automate the short term prediction of patient arrivals. It uses a multi-stage data aggregation scheme to deal with problems that may arise from limited arrival observations, an analysis phase to determine the existence of trends and seasonality, and an optimisation phase to determine the most appropriate smoothing method and the optimal parameters for this method.

The arrival forecasts for future time periods are used in conjunction with a simple demand modelling method based on a revised stationary independent period by period approximation queueing algorithm to determine the staff levels needed to service the likely arrivals and then determines a probability of bypass based on a comparison of required and available resources.

1 Introduction

This paper describes a system designed to be part of the process for managing Emergency Department (ED) bypass. An ED is on bypass when it has to turn away ambulances, typically because all cubicles are full and there is no opportunity to move patients to other beds in the hospital, or because the clinicians on duty are fully occupied dealing with critical patients who require individual care.

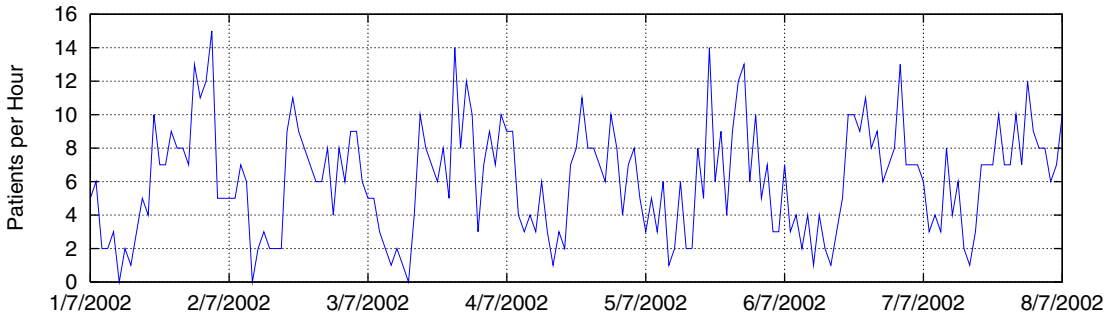
Bypass management is part of the more general bed management and admission–discharge procedures in a hospital. However, a very important part of determining the likelihood of bypass occurring in the near future, typically the next 1, 4 or 8 hours, is the ability to predict the probable patient arrivals, and then, given the current workload and staff levels, the probability that there will be sufficient resources to deal with these arrivals.

Here, we consider the implementation of a multi-stage forecasting method [1] to predict patient arrivals, and a demand management queueing method [2], to assess the likelihood of ED bypass.

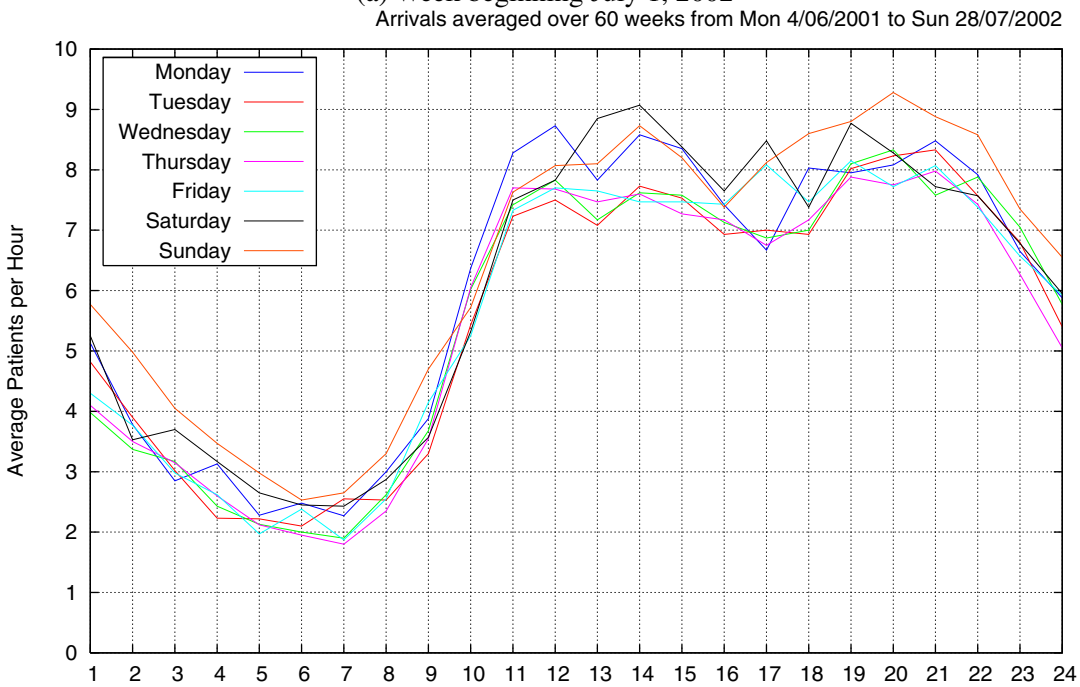
The prototype computer program implementing the method has been designed to run on a hospital intranet and to extract patient arrival data from hospital patient admission and ED databases. The program incorporates a range of exponential smoothing procedures. A user can specify the particular smoothing procedure for a data set or to configure the program to automatically determine the best procedure from those available and then use that method.

For the results presented here, we configured the program to automatically find the best smoothing method since this is the way it is likely to be used in an ED where the staff are more concerned with treating patients than configuring forecast smoothing parameters.

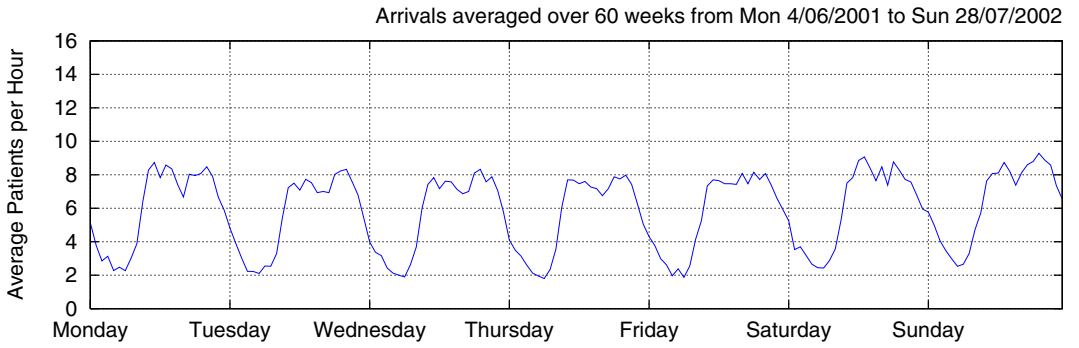
"Pictures" of "Arrivals" = Demand for Service



(a) Week beginning July 1, 2002



(b) Average by day of week



(c) Average weekly

Figure 1: Hourly patient arrivals, June 2001 to July 2002

For the optimisation we assume no a priori knowledge of the patient arrival patterns. The process involves simply fitting each of the nine different methods listed in Table 1 to the data, using the mean square fitting error, calculated using (3), as the objective function. The smoothing parameters for each method are all in (0, 1) and the parameter solution space is defined by a set of values obtained from an appropriately fine uniform discretization of this interval. The optimal values for each method are then obtained from a search of all possible combinations of the parameter values.

From the data aggregated at a daily level, repeat the procedure to extract data for each hour of the day to form 24 time series (12am–1am, 1am–2am, . . ., 11pm–12am). Apply the selected smoothing method, or the optimisation algorithm, to each time series and generate forecasting data for those future times of day within the requested forecast horizon. The forecast data generated for each time of day are scaled uniformly in each day in order to match the forecasts generated from the previously scaled daily data.

Output: Display the historical and forecasted data for each of the sets of aggregated observations constructed during the initialisation phase.

The generalisation of these stages is straightforward. For example, if the data was aggregated to a four-weekly (monthly) level, then the first scaling step would be to extract the observations from the weekly data to form four time series, corresponding to the first, second, third and fourth week of each month. Historical data at timescales of less than one day are scaled to the daily forecasts. For example, observations at a half-hourly timescale are used to form 48 time series for scaling to the day forecasts.

4.3 Output from the multi-stage method

Figures 2 and 3 show some of the results obtained from using the multi-stage forecasting method to predict ED arrivals using the 60 weeks of patient arrival data described in Section3. The forecasted data were generated from an optimisation that used the multi-stage forecasting method to minimise the residuals of (3) across all the smoothing methods in Table 1.

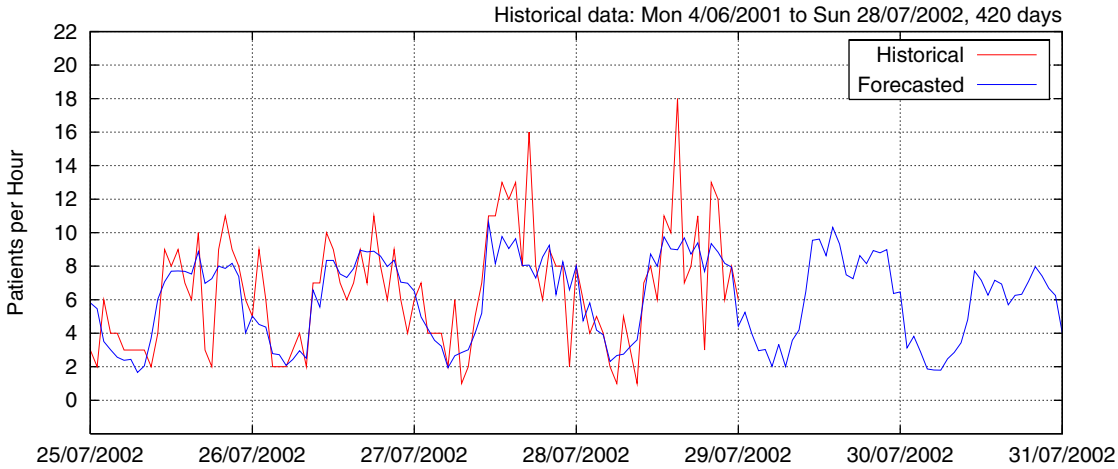


Figure 2: Hourly historical and forecasted data 25/7/2002–31/7/2002

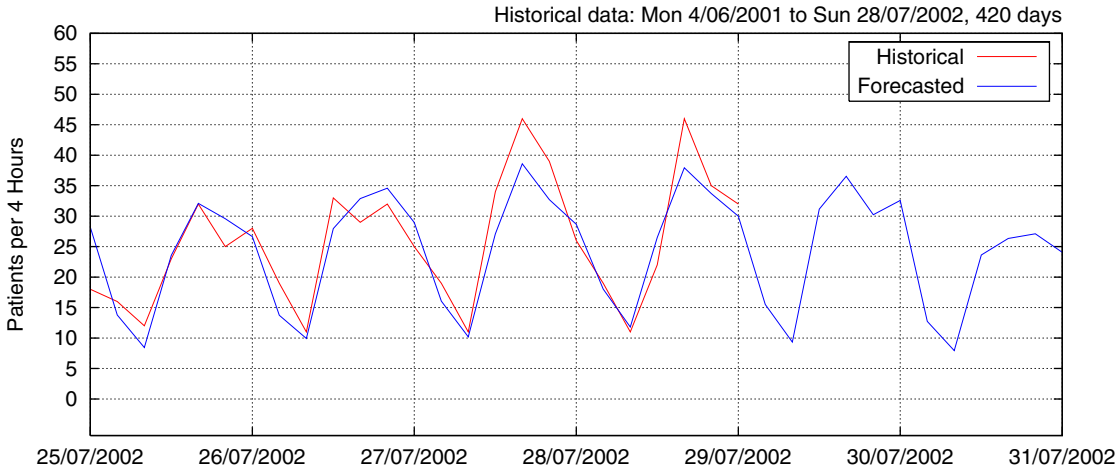
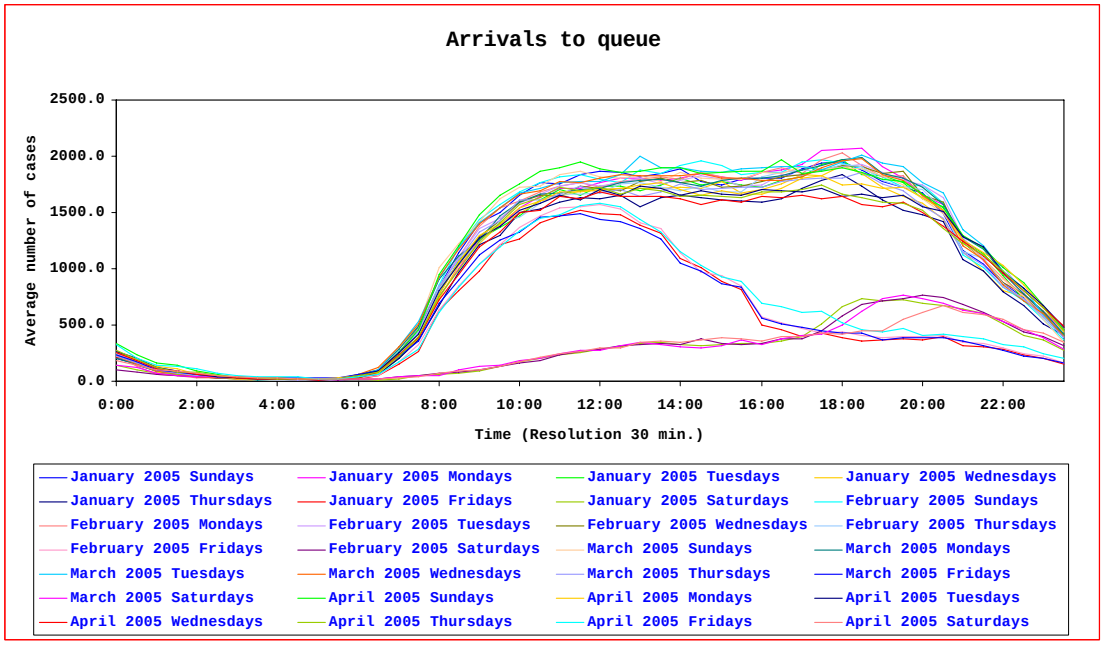


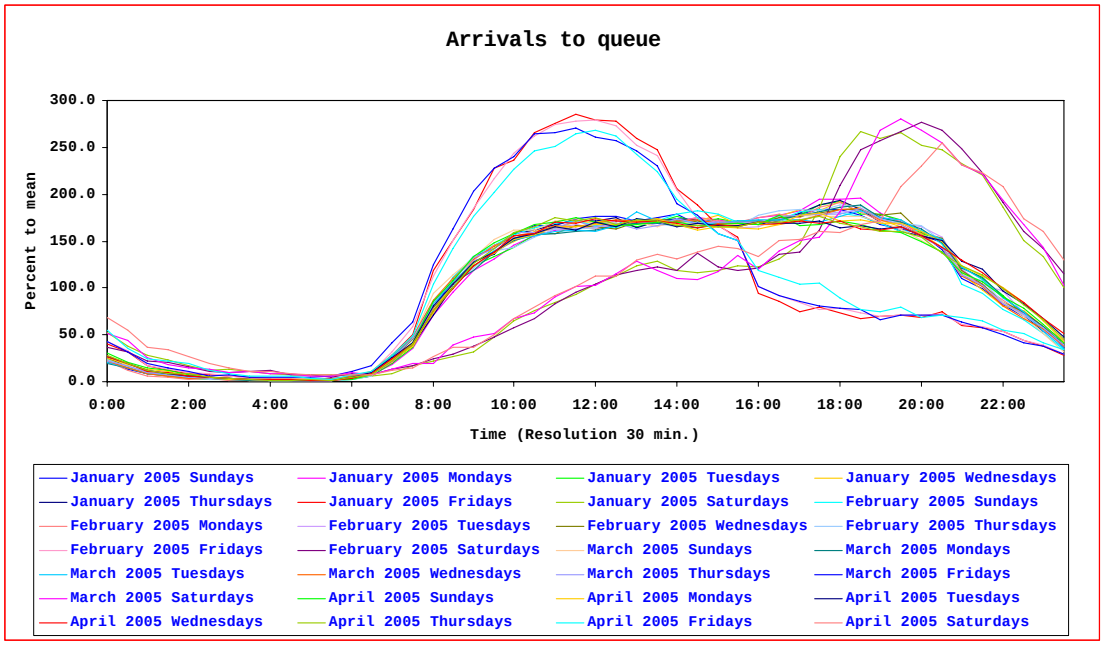
Figure 3: Four-hourly historical and forecasted data 25/7/2002–31/7/2002

Averaging (30 min) : over time
+ Aggregation (ALL Sundays) : own days

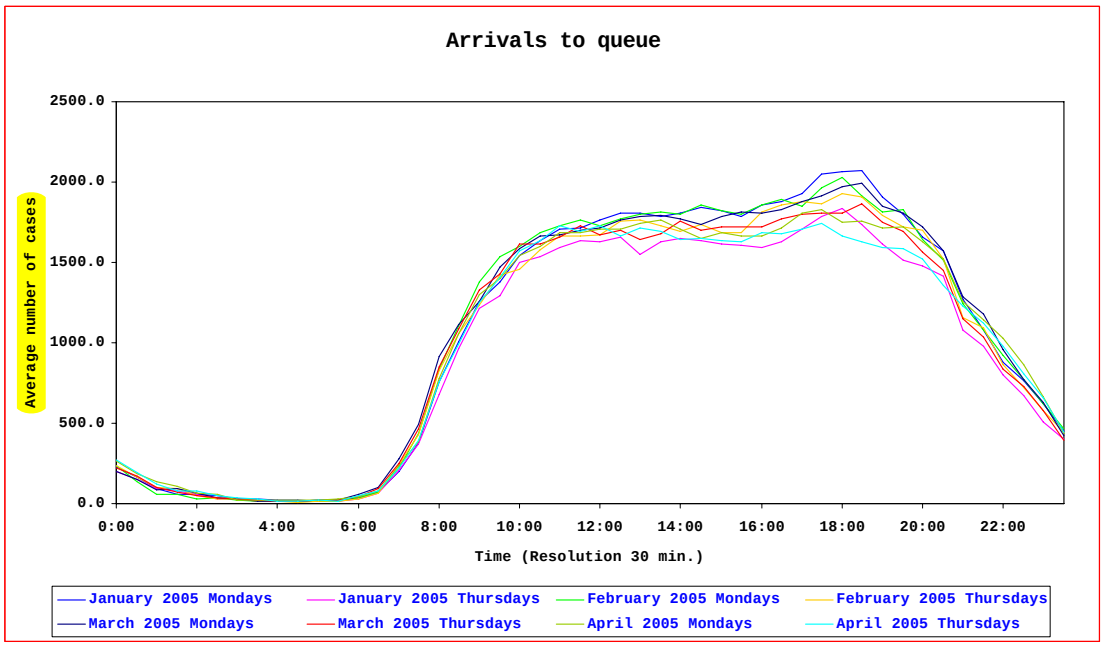
Arrival Patterns, Israeli Telecom, 2005

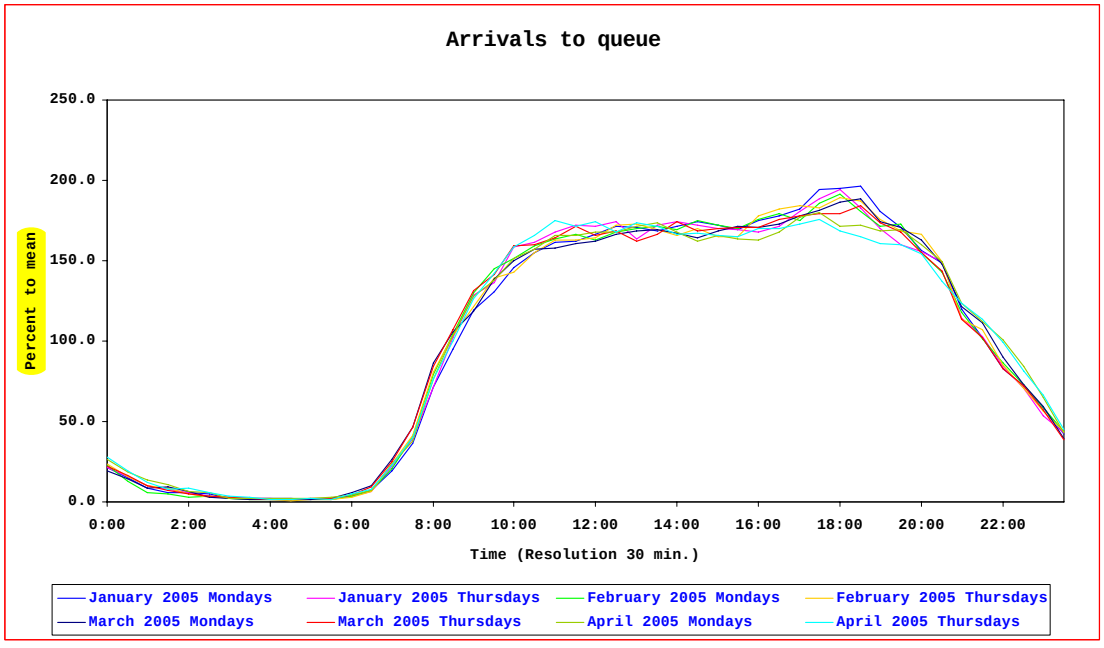


% to Mean

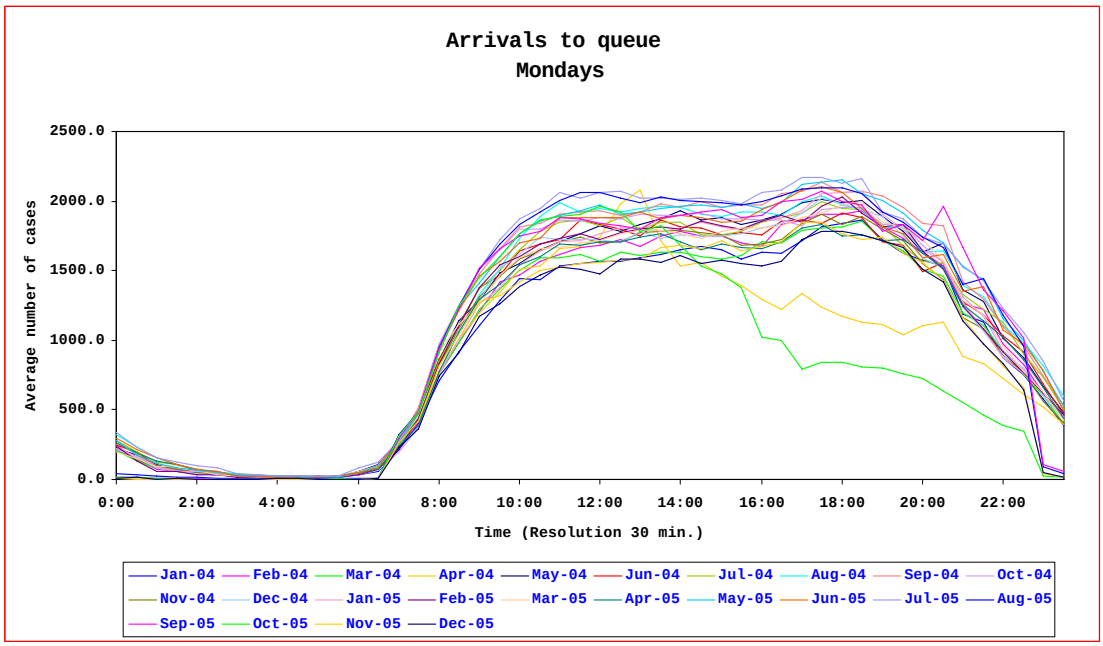


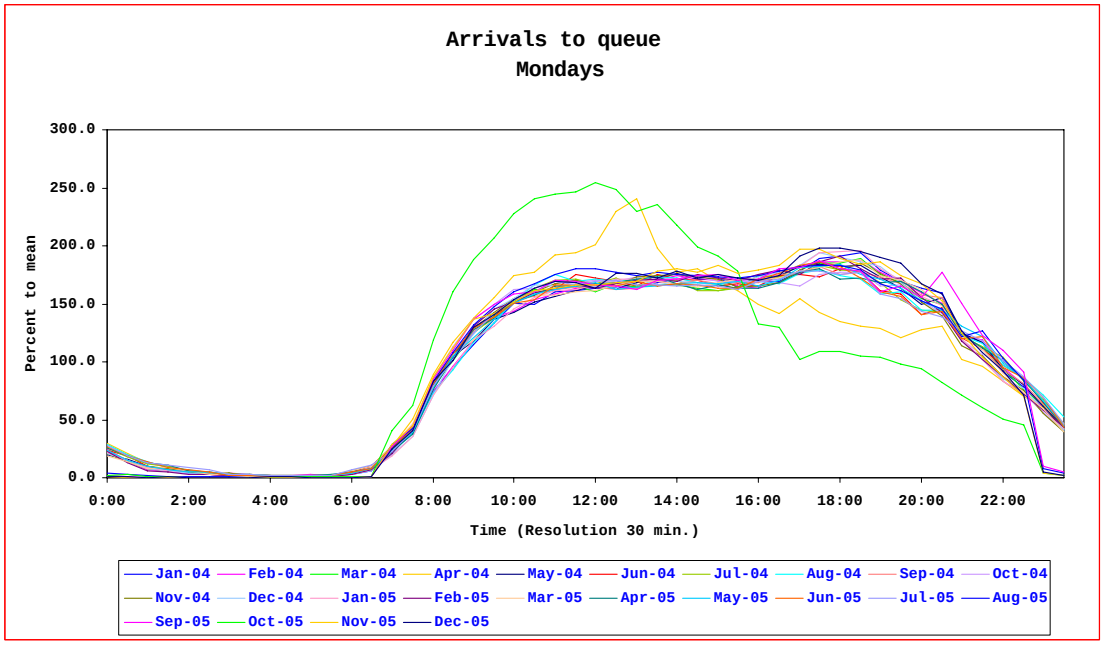
Mondays (Busiest) and Thursdays (Lightest), 2005



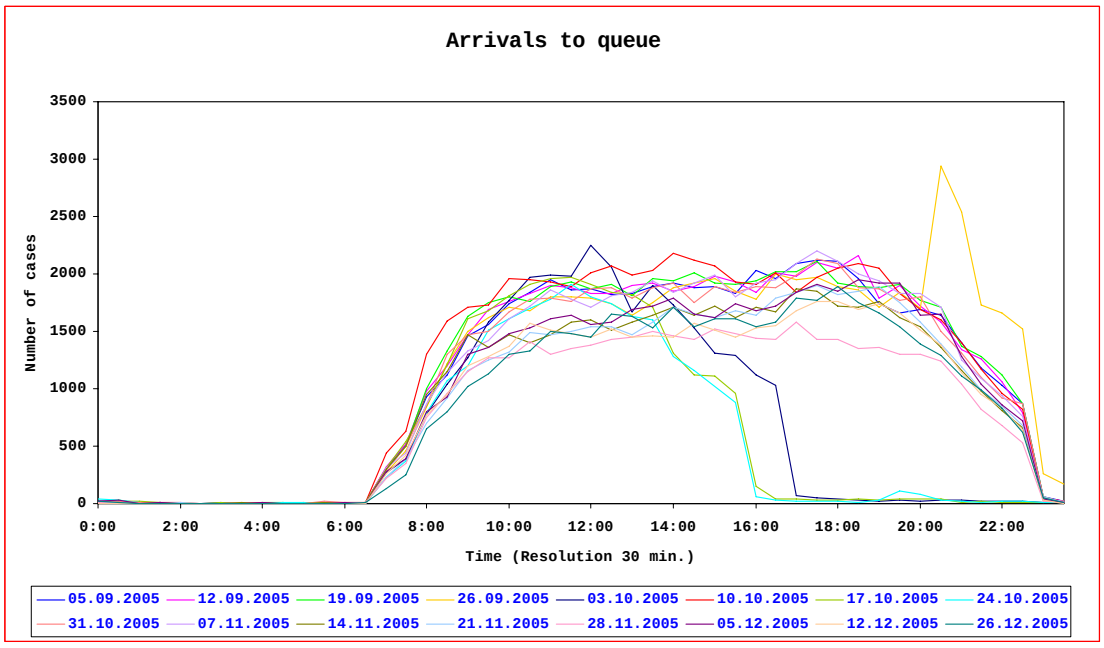


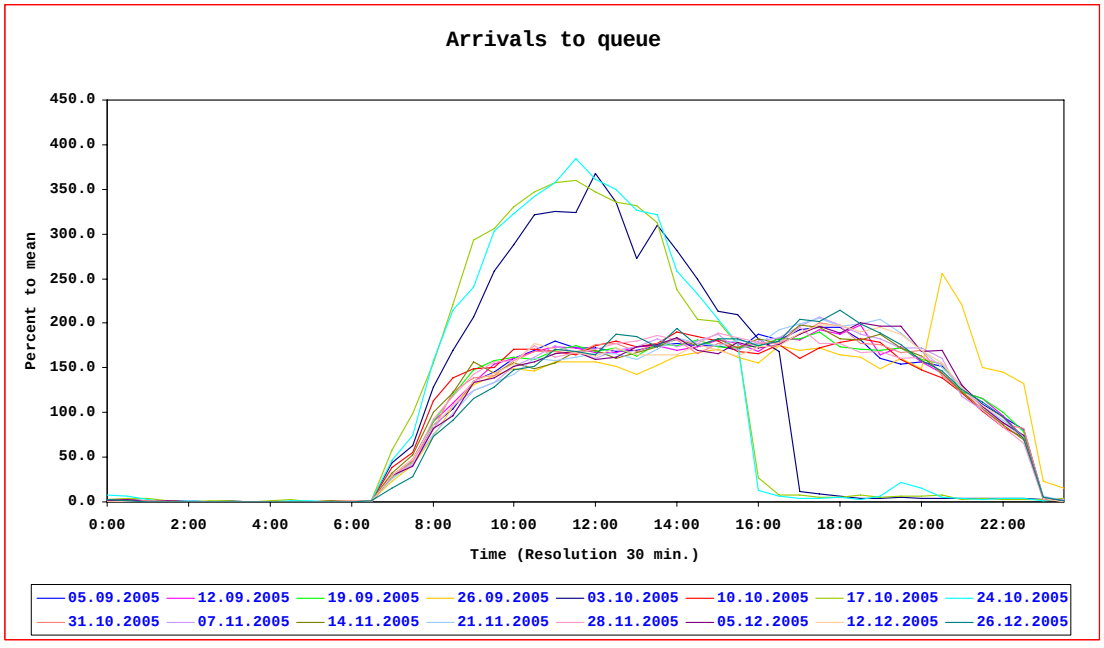
Mondays, 2004-5 (Averages)





Mondays, 2005 (Individual Days)

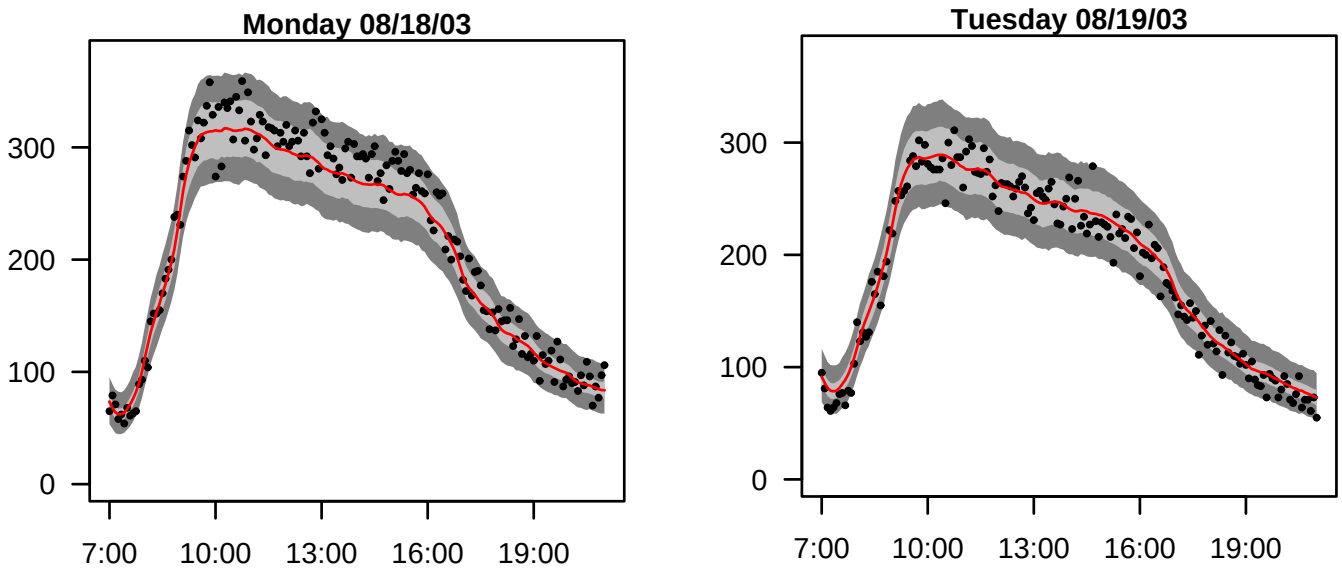




Why Model Arrivals? (Model, Building-Block)

Predict Forecast Performance: Example

US Bank: Forecast "Performance" (Weinberg, Brown, Stroud, 2005)



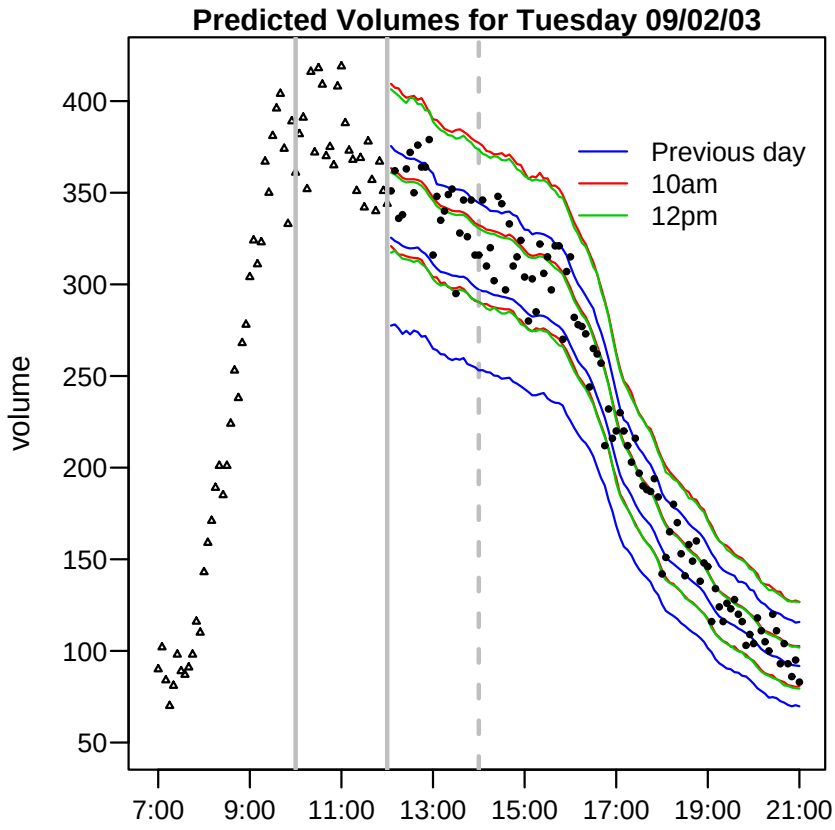
Wider confidence intervals for **number of calls**.

Narrower confidence intervals for **arrival rate** (Poisson parameter).

Note: **staffing** models require an **arrival rate** as input.

Within-Day Updating

Comparison between Day-Ahead and Within-Day Predictions (Weinberg, Brown, Stroud, 2005)



Conclusion: Morning information is important but no significant difference between 10am and 12am.

Submitted to the Annals of Applied Statistics

THE BEST LINEAR UNBIASED ESTIMATOR FOR CONTINUATION OF A FUNCTION

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We show how to construct the best linear unbiased predictor (BLUP) for the continuation of a curve in a spline-function model. We assume that the entire curve is drawn from some smooth random process and that the curve is given up to some cut point. We demonstrate how to compute the BLUP efficiently. Confidence bands for the BLUP are discussed. Finally, we apply the proposed BLUP to real-world call center data. Specifically, we forecast the continuation of both the call arrival counts and the workload process at the call center of a commercial bank.

1. Introduction. Many data sets consist of a finite number of multi-dimensional observations, where each of these observations is sampled from some underlying smoothed curve. In such cases it can be advantageous to address the observations as functional data rather than as multiple series of data points. This approach was found useful, for example, in noise reduction, missing data handling, and in producing robust estimations (see the books Ramsay and Silverman, 2002, 2005, for a comprehensive treatment of functional data analysis). In this work we consider the problem of forecasting the continuation of a curve using functional data techniques.

The problem we consider here is relevant to longitudinal data sets, in which each observation consists of a series of measurements over time that describe an underlying curve. Examples of such curves are growth curves of different individuals and arrival rates of calls to a call center or of patients to an emergency room during different days. We assume that such curves, or measurement series that approximate these curves, were collected previously. We would like to estimate the continuation of a new curve given its beginning, using the behavior of the previously collected curves.

Although each observation consists of a finite number of points, the observation can be thought of as a smooth function. This dual representation leads to two different approaches when attempting to solve the prediction problem. In the discrete approach, each observation is a longitudinal vector of length $p+q$. We are interested in the prediction of the last q-length part

Keywords and phrases: functional data analysis, best linear unbiased predictor, call center data, B-splines

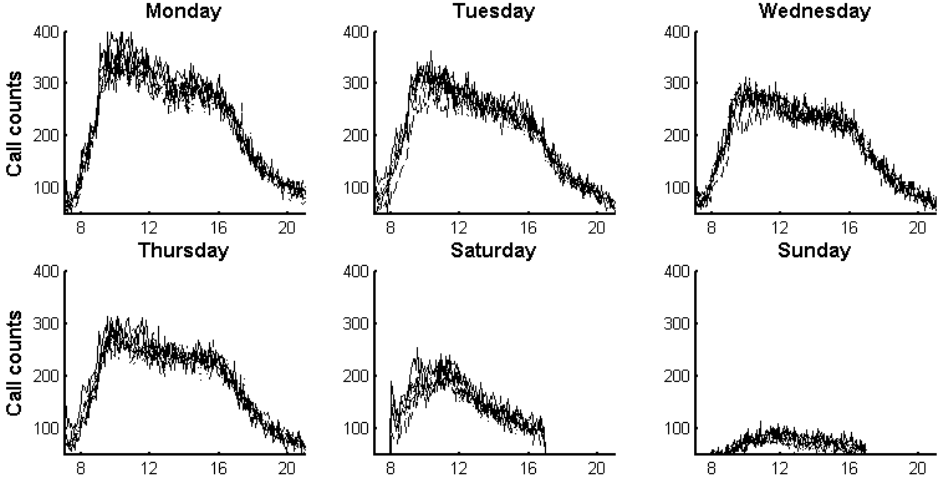


FIG 1. *Arrival count in five-minutes resolution for six successive weeks, grouped according to weekday (Friday was omitted due to space constraints). There is a clear difference between workdays, Saturdays, and Sundays. For the working days, it seems that there is some common pattern. Between 7 AM and 10 AM the call count rises sharply to its peak. Then it decreases gradually until 4 PM. From 4 PM to 5 PM there is a rapid decrease followed by a more gradual decrease from 5 PM until 12 AM. The call counts are smaller for Saturday and much smaller for Sunday. Note also that the main activity hours for weekends are 8 AM to 5 PM, as expected.*

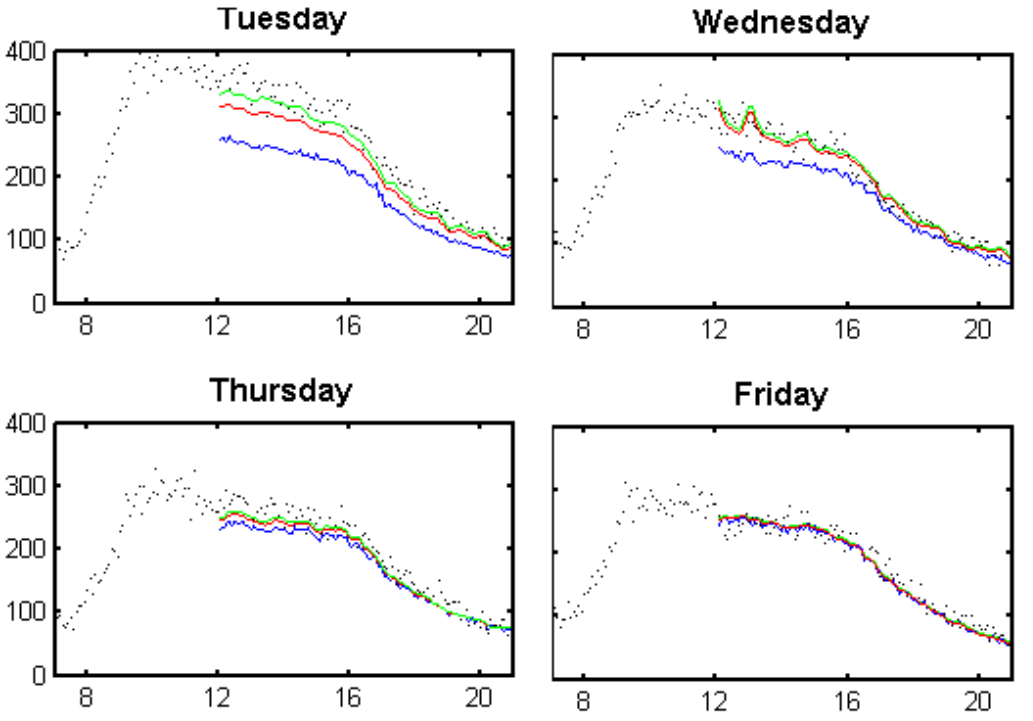


FIG 2. *Forecasting results for the week following Labor Day (Sept. 2-5, 2003) for the call arrival process of the first example. Labor Day itself (Monday) does not appear since holiday data is not included in the data set. The black dots represent the true call counts in five-minutes resolution. The forecasts based on previous days, 10 AM data, and 12 PM data are represented by the blue, red, and green lines, respectively.*

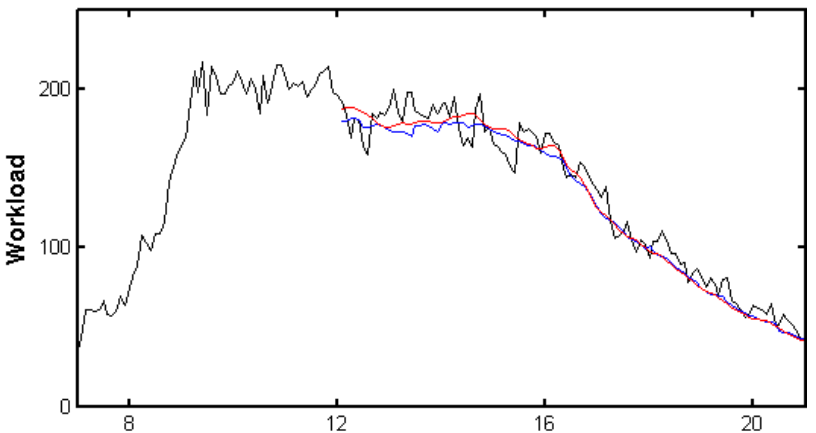


FIG 3. *Workload forecasting for Friday, September 5, 2003, using both the direct and the indirect methods. The black curve represents the workload process estimated after observing the data gathered throughout the day. The blue and red curves represents the workload forecasts for the indirect and direct forecasts, respectively, given data up to 12 PM.*

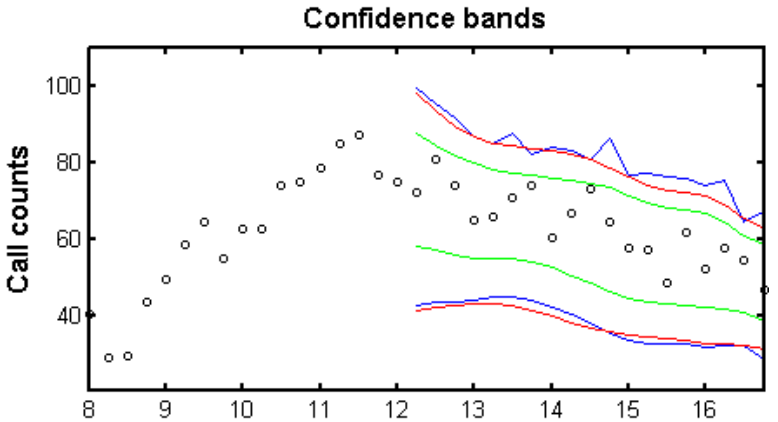


FIG 4. *Confidence bands for Sunday, August 10, 2003. The black dots represent the true call counts in fifteen-minutes resolution. The confidence bands based on previous days, 10 AM data, and 12 PM data are represented by the blue, red, and green lines, respectively.*

Recall: 4 Constructions of the Poisson Process

Interarrival times: Exponential iid; for Simulations.

Probability-of-arrival during small intervals:
Counting & Levy (stationary independent increments),
with the properties:

$$\begin{aligned} P\{\ A(t+dt)-A(t)=1\} &= \lambda dt + o(dt), \\ \{\ \qquad\qquad\quad =0\} &= 1-\lambda dt + o(dt), \\ \{\ \qquad\qquad\quad \geq 2\} &= o(dt). \end{aligned}$$

Axiomatic: Counting & Levy suffices!
(But up to λ).

Intuitive: from Bernoulli to Poisson
(The Law of Rare Events).

Intuitive Construction (Animation): from Bernoulli to Poisson

Model for "completely **random**" arrivals, over the time interval $[0, T]$, at rate λ:

- **Large** number of customers n, each one calling during $[0, T]$, with a **small** probability $p_n \approx \frac{\lambda T}{n}$ (rate λ).

- Times of calls **uniformly** distributed over $[0, T]$.

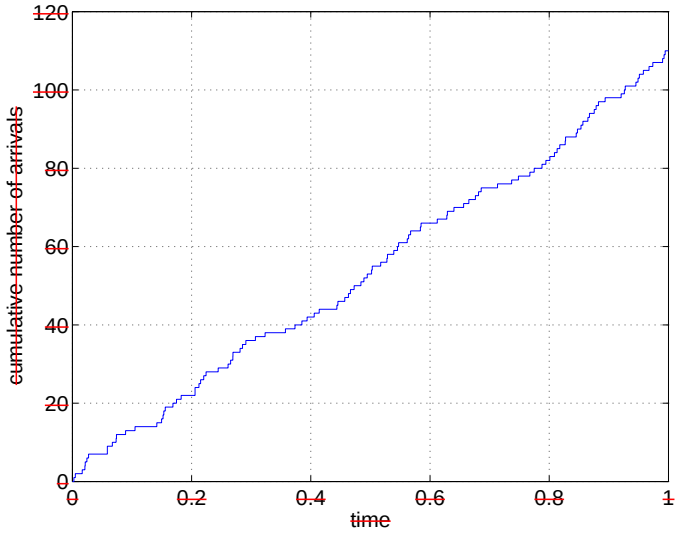
- Then: number of calls $A(T) \stackrel{d}{=} \text{Bin}(n, p_n)$.

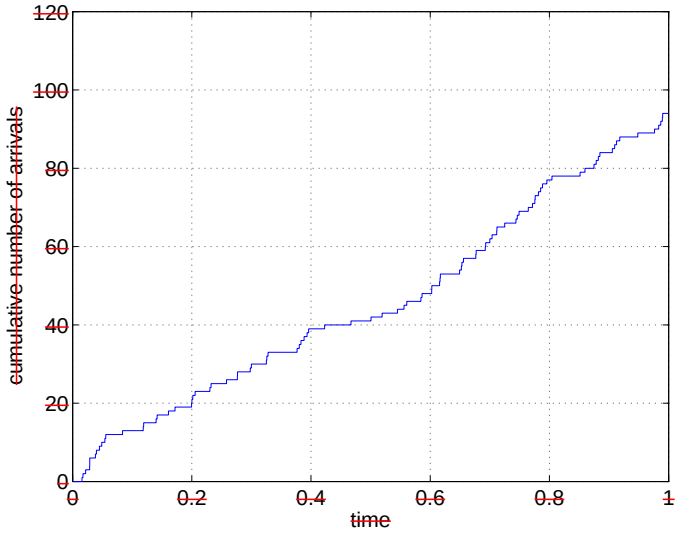
- Note: $np_n \Rightarrow \lambda T$, as $n \Rightarrow \infty$.

- By **Law of Rare Events**: $A(T) \Rightarrow \text{Poiss}(\lambda T)$.

Simulation Examples (Mathlab)

$n = 10000$, $p_n = 0.01$ **$n = 100000$, $p_n = 0.001$**





Hall, Chapter 3: **The Arrival Process** $N = \{N(t),\ t \geq 0\}$

§3.1 *Definition* 3.2 requires too much. As discussed, Levy + counting $\Rightarrow$ $\exists \lambda > 0 \ni N(t) - N(s) \sim$ Poisson $[\lambda(t-s)]$.

In particular,

$$\begin{array}{rrl} P\{ & N(t+dt) - N(t) = 1\} & = \lambda dt + o(t) \\ \{ & = 0\} & = 1 - \lambda dt + o(t). \\ \{ & > 1\} & = o(t) \end{array}$$

§3.2 *Derivation* of the Poisson distribution from Bernoulli.

§3.3 *Properties* of the Poisson Process.

1. Poisson *marginals*; number of events in any interval is Poisson;

$$\begin{aligned} EN_t &= \lambda t\,, \text{Var } N_t = \lambda t \\ &\Rightarrow C = \frac{\sigma}{E} = \frac{\sqrt{\lambda t}}{\lambda t} = \frac{1}{\sqrt{\lambda t}} \text{ small for } t \text{ large.} \end{aligned}$$

2. *Interarrival times* which are iid exp (λ).
   Beginning of proof: $P(T_1 \geq t) = P(N_t = 0) = e^{-\lambda t},\ t \geq 0$.
   This is a characterizing property that is practical for simulation.
   Extensions to $T_2, T_3, \ldots,$ and their independence, if rigorous, requires more than the "it should be apparent" in Hall, pg. 58.
3. *Memoryless* property: time till next event does not depend on the elapsed time since the last event.
4. $S_n = T_1 + \cdots + T_n \sim$ Gamma (n, λ) = Erlang.
5. *Order-statistics* property: Given $N(t) = n$, the unordered event times are distributed as n iid r.v., uniformly distributed on $[0, t]$.

   $\Rightarrow$ simulation over $[0,t]$: $N(t) \sim$ Poisson (λt); $U_1, U_2, \ldots, U_{N(t)}$ iid $U[0,t]$.

§3.4 *Goodness* of *Fit*

How well does a Poisson model fit our arrival process?

Qualitative assessments:

Airplanes landing times at a single runway, during an hour:	no
Airplanes landing times at a large airport, during an hour:	plausible
Job candidates that arrive at their appointments during an hour:	no
Visits to a zoo, most of which arrive in groups, during an hour:	no
Arrival times at a bank ATM = **A**utomatic **T**eller **M**achine, during an hour:	plausible

§3.5 *Quantitative Tests*

Graphical Tests:

cumulative arrivals vs. a straight line (Fig. 3.2)

paired successive interarrivals (Fig. 3.4)

exponential interarrivals
(How do you identify exp $(\cdot)$ when you see one? Use Histograms!)

§3.6 *Parameter Estimation*

Estimate λ = arrival rate.

MLE (Max. Likelihood Estimator), given $A(t),\ t \leq T : \hat{\lambda} = \frac{A(T)}{T}$.

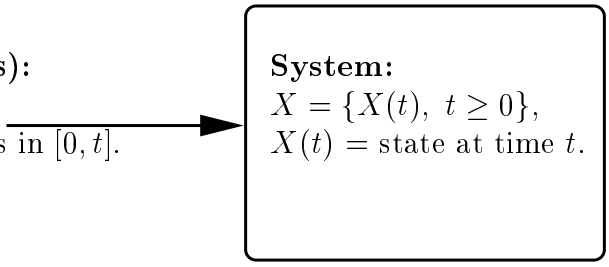
Confidence intervals for $\frac{1}{\lambda} : \frac{T}{A(T)} \pm z_\alpha \frac{T}{A(T)^{3/2}}$ (3.34)

Sample-size: for $(1-\alpha)$-confidence interval of width w, $N \geq [\frac{2z_\alpha}{w\lambda}]^2$.

Thus, for $w = \epsilon \cdot \frac{1}{\lambda}$, we need $N \geq [\frac{2z_\alpha}{\epsilon}]^2$.
(Eg.: 95%-confidence interval of width = 10% of mean, requires $N \geq [\frac{2\times 1.96}{0.1}]^2 \approx$ 1500!)

PASTA = Poisson Arrivals See Time Averages

A rigorous proof in the following general setting was first presented by R.Wolff.



Time average $\quad \bar{\tau} \triangleq \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T X(t)dt$

Customer average $\quad \bar{c} \triangleq \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} X(S_n-)$

where S_n = n-th arrival time.

PASTA assumptions:

(i) A is **Poisson**, and

(ii) **Lack of Anticipation.** For every $t \geq 0$, $\{A(t+u) - A(t) : u \geq 0\}$ is independent of $\{X(s) : 0 \leq s \leq t\}$.

Then $\bar{\tau} = \bar{c}$, in the following precise sense:
If one limit exists, then the other exists as well, in which case they are equal.

Second version: Let B be an arbitrary subset of a process X. Define by

$$\bar{\tau}_B \triangleq \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T I\{X(t) \in B\}dt$$

the fraction of time that the system spends in B, and let

$$\bar{c}_B \triangleq \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} I\{X(S_n-) \in B\}$$

be the fraction of arrivals that find ("see") the system in B.
Then $\bar{\tau}_B = \bar{c}_B$ in the same sense as above.

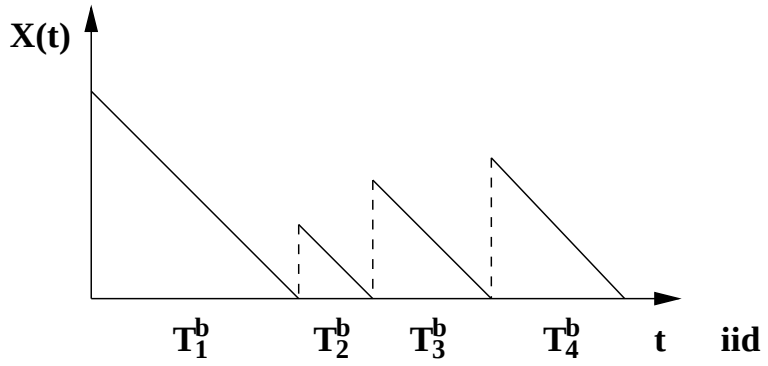
Application of **PASTA:** **Biased Sampling**

A *renewal process* is a counting process with iid interarrivals.

Descriptions:	$R = \{R(t),\ t \geq 0\}$ or	$\{T_1, T_2, \ldots\}$ iid, or	$\{S_1, S_2, \ldots\}$
Example:	Poisson	exponential	Erlang

Story: Buses arrive to a bus stop according to a renewal process $R_b = \{R_b(t), t \geq 0\}$.
T_i^b — times between arrivals of the buses.
Passengers arrive to the bus stop in a completely random fashion (Poisson).
S_i^p — arrival times of the passengers.

Question: How long, on average, do they wait? Plan service-level.



$A = \{A(t),\ t \geq 0\}$ = Poisson arrivals of passengers.

$X = \{X(t),\ t \geq 0\}$ = state = *Virtual waiting time.*

PASTA: $\lim\limits_{T\uparrow\infty} \dfrac{1}{T} \int_0^T X(t)dt = \lim\limits_{N\uparrow\infty} \dfrac{1}{N} \sum\limits_{n=1}^{N} X(S_n^p-) = \bar{\tau}$

$$\begin{aligned}
\Rightarrow \bar{\tau} &= \frac{1}{T} \cdot \text{ (area under } X, \text{ over } [0,T]) \\
&\approx \frac{1}{T} \cdot \left(\frac{1}{2}(T_1^b)^2 + \frac{1}{2}(T_2^b)^2 + \cdots + \frac{1}{2}(T_{R_b(T)}^b)^2 \right) \\
&= \frac{R_b(T)}{T} \cdot \frac{1}{2} \cdot \frac{T_1^2 + \cdots + T_{R_b(T)}^2}{R_b(T)} \xrightarrow[T\uparrow\infty]{} \frac{1}{E(T_1^b)} \cdot \frac{1}{2} \cdot E(T_1^b)^2 \text{, by SLLN} \\
&= \underbrace{\frac{1}{2}E(T_1^b)}_{\text{“Deterministic” answer}} \quad \underbrace{[1 + c^2(T_1^b)]}_{\text{Bias, due to variability}}, \qquad c = \frac{\sigma}{E} \text{ coefficient of variation.}
\end{aligned}$$

Check Poisson bus arrivals to derive Paradox:
1(“stochastic” answer) = $\frac{1}{2}$ (“deterministic” answer).

Time-Inhomogeneous Poisson Process

Counting process with independent increments:

$$\begin{aligned} P\{\ A(t+dt)-A(t)=1\} &= \lambda(t)dt+o(dt),\\ \{\ \ =0\} &= 1-\lambda(t)dt+o(dt),\\ \{\ \ >1\} &= o(dt). \end{aligned}$$

Main Property:

Poisson number of arrivals over intervals:

$$A(T_2)-A(T_1) \ \stackrel{d}{=}\ \text{Poiss}\left(\int_{T_1}^{T_2}\lambda(s)ds\right).$$

Construction from time-homogeneous:

(Time-Change in Stochastic Processes; Thinning here)

~~Data: Arrival rate $\lambda(t)$, $0 \le t \le T$.~~

~~Let $\lambda_{max} = \max_{t\in[0,T]} \lambda(t)$.~~

~~**1.** Simulate a homogeneous Poisson(λ_{max}) process.~~

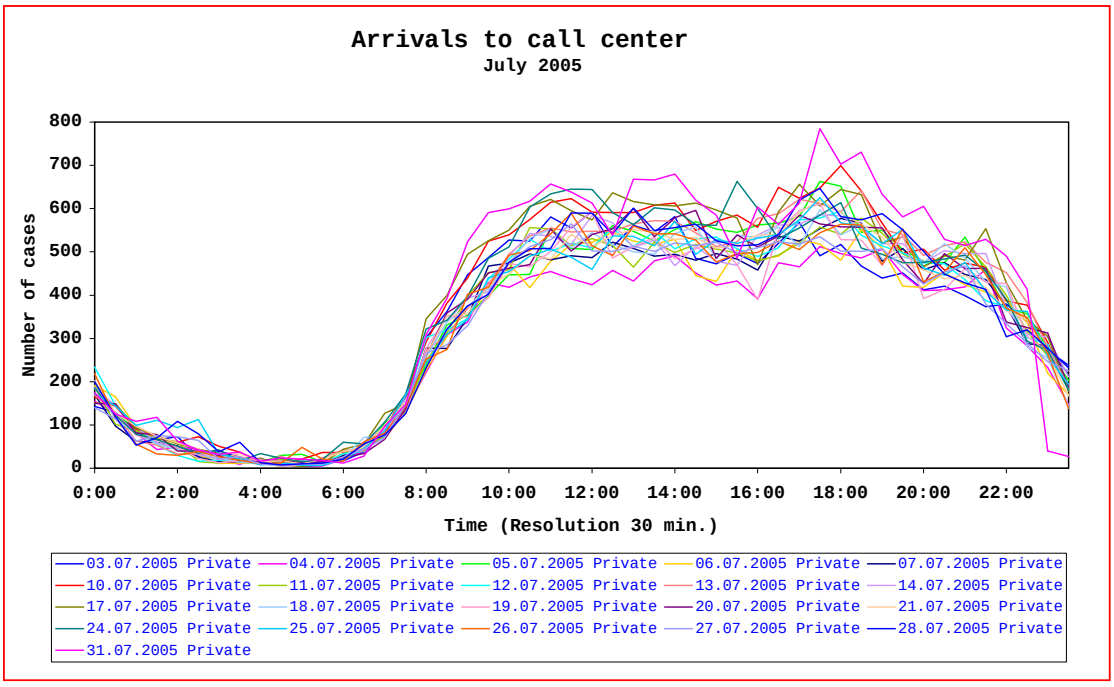
~~**2.Thinning.** For each arrival S_i generate $U_i \stackrel{d}{=} U(0,1)$.~~

~~Let $p_i = \lambda(S_i)/\lambda_{max}$.~~

~~$U_i \le p_i$, accept arrival;~~

~~$U_i > p_i$, reject arrival.~~

Arrivals to a Call Center: How to Model?



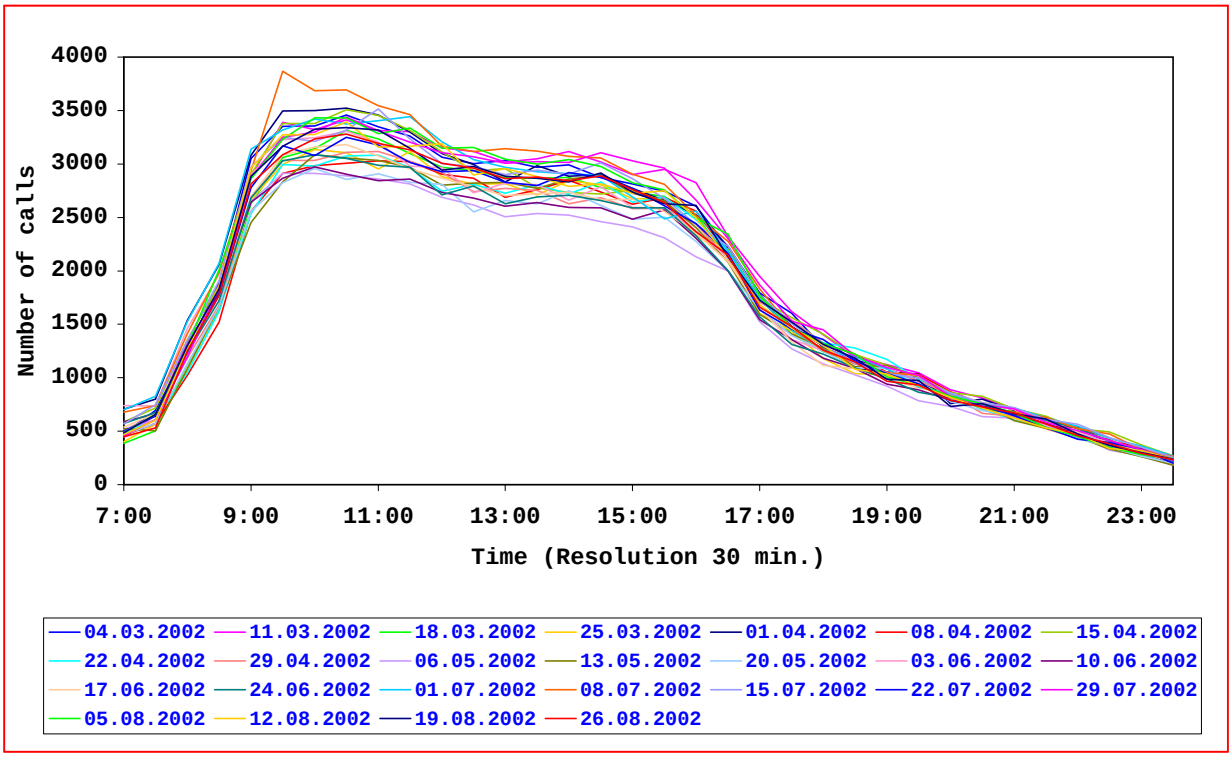
- Arrivals over the day are **not** time-homogeneous.
- Arrivals over small intervals (15, 30, 60 min) are close to time-homogeneous Poisson.
- Arrivals over the day are non-homogeneous Poisson.

Practically: Test (Brown), then model, as a **Poisson process with piecewise-constant arrival rates**.

How to predict/forecast arrival rates?

Arrivals to a Call Center: Variability of the Arrival Rates

Number of Calls at a U.S. bank. Mondays. March 2002-August 2002.



25 Mondays overall.

- **13:00-13:30**: 25 observations, range: 2,500-3,2000; Sample Mean=2,842, BUT Sample Variance=24,539!
- **17:00-17:30**: Mean=1,705, Variance=10,356.

Conclude: **Number of calls during "similar" intervals not i.i.d Poisson**: **over-dispersion**.

A Test for Inhomogeneous Poisson Process

1. Break up the interval of a day into short blocks of time, say I (equal-length) blocks of length L.

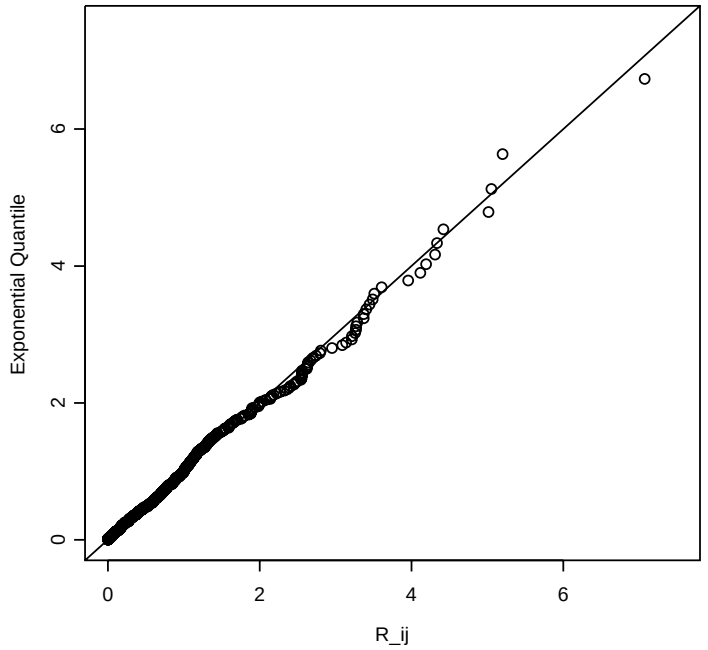
2. Let $T_{i0} = 0$ and
   T_{ij}: the j-th ordered arrival time in the i-th block, $i = 1, \ldots, I$ and $j = 1, ..., J(i)$,
   then define

$$R_{ij} = (J(i) + 1 - j) \left(- \log \left(\frac{L - T_{ij}}{L - T_{i,j-1}} \right) \right).$$

3. Under the null hypothesis that the arrival rate is constant within each given time interval, the $\{R_{ij}\}$ will be independent standard exponential variables.

4. Use any customary test for the exponential distribution; for example, Kolmogorov-Smirnov test.

Figure 3: Exponential (λ=1) Quantile plot for $\{R_{ij}\}$ from Regular calls (11:12am – 11:18am)



$L = 6$ min, $n = 420$, Kolmogorov-Smirnov statistic $K = 0.0316$ and the P-value is 0.2.

Forecasting Problem: Setup

Days are divided into time intervals, with an assumed constant arrival rate over an interval.

Practice: 15 min, 30 min, 1 hour.

N_{jk} = # of arrivals, during time-interval k, on day j.
Assume J days overall, with K intervals per day.

- **One-day-ahead** prediction:
  $N_{1,}, \ldots, N_{j-1,}$ known. Predict $N_{j1}, \ldots, N_{jK}$.
- **Several days (weeks) ahead** prediction.
- **Within-day** prediction.
  $N_{1,}, \ldots, N_{j-1,}, N_{j1}, \ldots, N_{j,k-1}$ known.
  Predict $N_{jk}, \ldots, N_{jK}$.

Practice: Do all the above, via nested rolling horizon (Weekly, Daily, Hourly).

Forecasting: Simple Methods

Most recent observation.

F_{jk} = most recent "similar" call volume.

Example: $F_{jk} = N_{j-7,k}$ (previous week).

Moving average.

Average of several (not too many) recent "similar" call volumes.

Most Recent, plus Yesterday's Correction.

Example: Factor accounting for a "busy yesterday".

What about sophisticated forecasting methods?
Active research.

Here, we shall compare the performance of simple methods against (given results of) sophisticated methods.

Forecasting: Goodness-of-Fit

N_{jk} = number of calls (day j, interval k);

F_{jk} = forecast.

Two ways to quantify forecasting accuracy:

1. Root Mean-Square Error (RMSE)

For each day j, calculate:

$$RMSE_j = \sqrt{\frac{1}{K}\sum_{k=1}^{K}(N_{jk} - F_{jk})^2}.$$

$$RMSE = \sum_{j=1}^{J}\frac{RMSE_j}{J}.$$

2. Average Percent Error (APE)

$$APE_j = \frac{100}{K}\cdot\sum_{k=1}^{K}\frac{|N_{jk} - F_{jk}|}{N_{jk}}.$$

$$APE = \sum_{j=1}^{J}\frac{APE_j}{J}.$$

Exogenous Arrivals to Service: How to Model?

- Axiomatically, "completely random arrivals" are **Poisson**.
- Arrivals over the day are not time-homogeneous.
- Hence, arrivals over the day are non-homogeneous Poisson.
- Arrivals over small intervals (15, 30, 60 min) are close to time-homogeneous Poisson.

Practically:
Test (L. Brown), then model, as a **Poisson process with piecewise-constant arrival rates**.

A (Common) Model for Call Arrivals

Whitt (99'), Brown et. al. (05'), Gans et. al. (09'), and others:

Doubly-stochastic (Cox, Mixed) Poisson with instantaneous rate

$$\mathbf{\Lambda(t) = \lambda(t) \cdot X} \ ,$$

where $\int_0^T \lambda(t)dt = 1$.

- $\boldsymbol{\lambda(t)}$ = "Shape" of weekday [**Predictable variability**]
- **X** = Total # arrivals [**Unpredictable variability**]

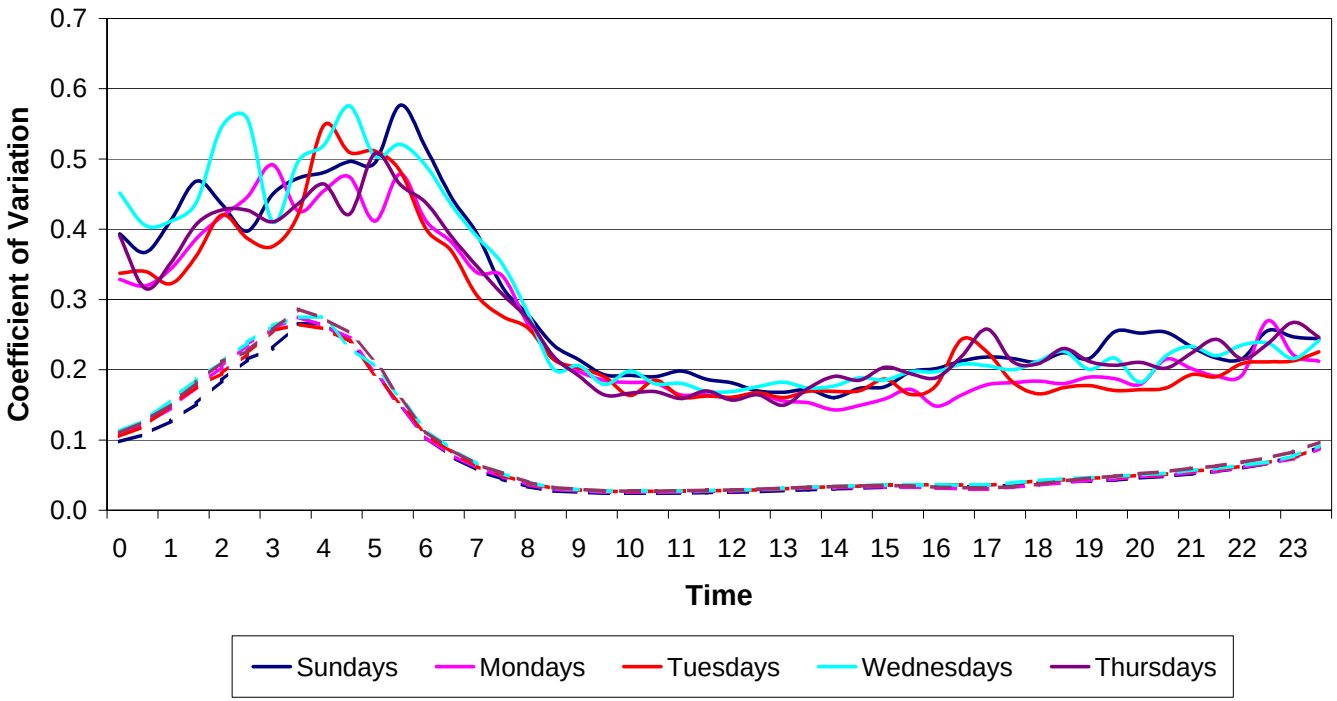
w/ Maman & Zeltyn (09'):
Above assumes **"too-much" stochastic variability!**

Over-Dispersion (Relative to Poisson), Maman et al. ('09)

Israeli-Bank Call-Center

Arrival Counts - Coefficient of Variation (CV), per 30 min.

Sampled CV - solid line, **Poisson** CV - dashed line

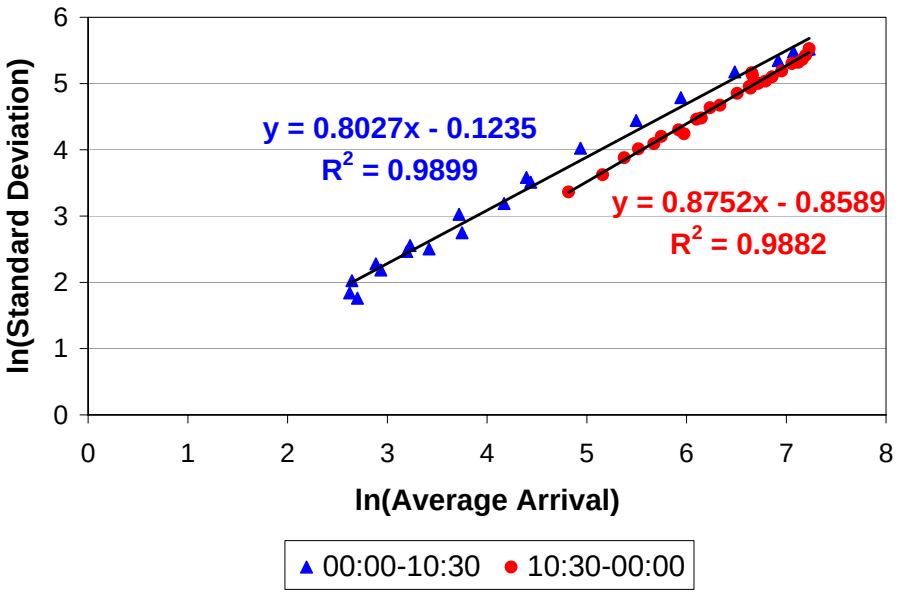


- 263 regular **days**, 4/2007 - 3/2008.
- Poisson CV $= 1/\sqrt{\text{mean arrival-rate}}$.
- Sampled CV's $\gg$ Poisson CV's $\Rightarrow$ **Over-Dispersion**.

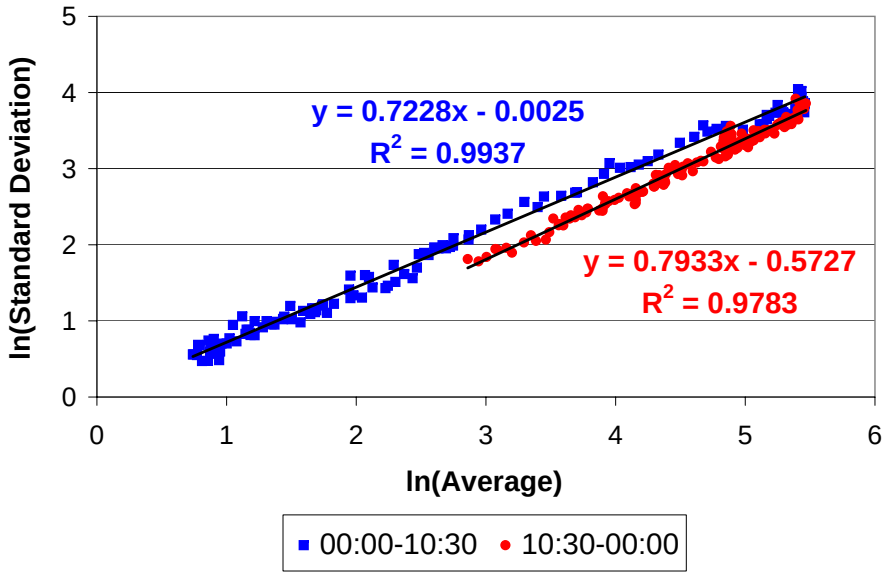
Over-Dispersion: Fitting a Regression Model

ln(STD) vs. ln(AVG)

Tue-Wed, 30 min resolution



Tue-Wed, 5 min resolution



Significant linear relations (Aldor & Feigin):

$$\ln(\text{STD}) = c \cdot \ln(\text{AVG}) + a$$

Over-Dispersion: Random Arrival-Rate Model

The **linear relation** between ln(STD) and ln(AVG) motivates the following model:

Arrivals distributed **Poisson with a Random Rate**

$$\Lambda = \lambda + \lambda^{c} \cdot X, \quad 0 \leq c \leq 1 ;$$

- X is a random-variable with $E[X] = 0$, capturing the magnitude of **stochastic deviation** from mean arrival-rate.
- c determines **scale-order** of the over-dispersion:
  $c = 1$, proportional to λ;
  $c = 0$, Poisson-level, same as $0 \leq c \leq 1/2$.

In **call centers**, over-dispersion (per 30 min.) is of order λ^{c}, $c \approx 0.8 - 0.85$.

Over-Dispersion: Distribution of X ?

- Fitting a **Gamma Poisson** mixture model to the data:
  Assume a (conjugate) prior Gamma distribution for the arrival rate $\Lambda \stackrel{d}{=} Gamma(a, b)$.
  Then, $Y \stackrel{d}{=} Poiss(\Lambda)$ is Negative Binomial.

- Very good fit of the Gamma Poisson mixture model, to data of the Israeli Call Center, for the majority of time intervals .

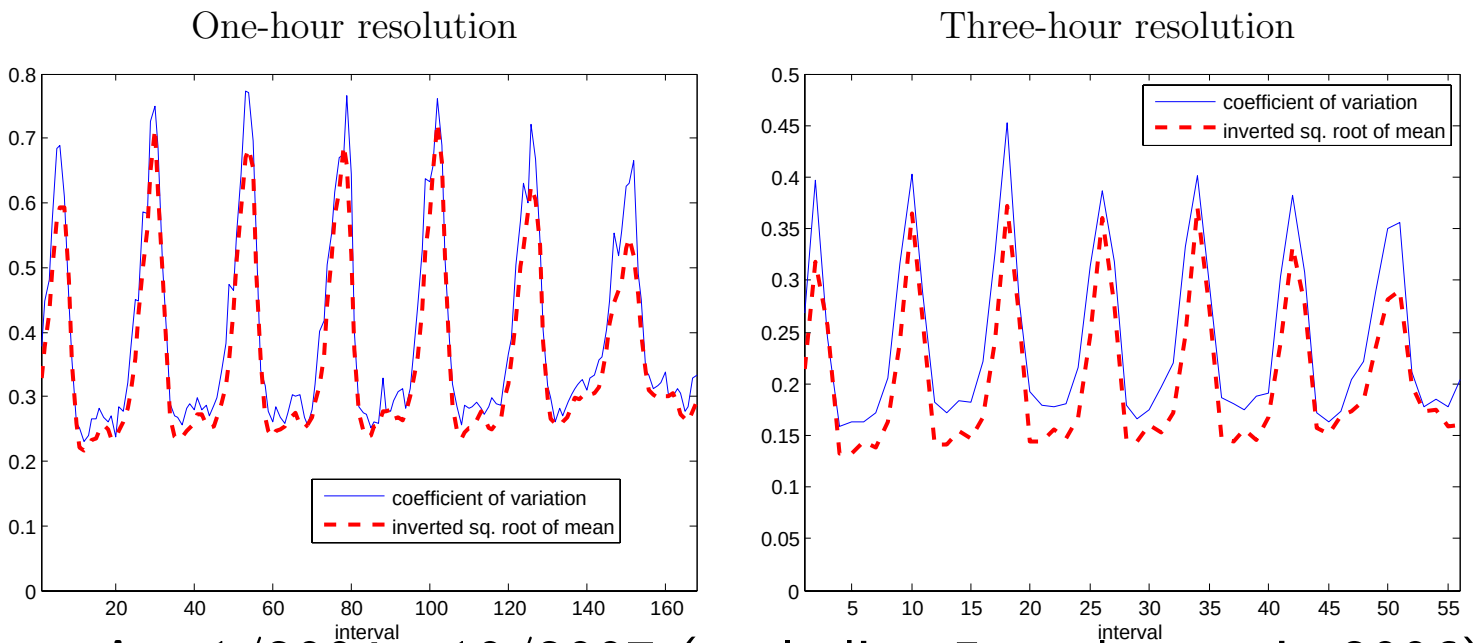
- Relation between our c-based model and Gamma-Poisson mixture is established.

- Distribution of X derived, under the Gamma prior assumption: X **is asymptotically normal**, as $\lambda \to \infty$.

Over-Dispersion: The Case of ED's

Israeli-Hospital Emergency-Department

Arrival Counts - Coefficient of Variation, per 1-hr. & 3-hr.



- 194 **weeks**, 1/2004 - 10/2007 (excluding 5 weeks war in 2006).
- Moderate over-dispersion: **c = 0.5** reasonable for hourly resolution.
- **ED beds in conventional QED** (**Less** var. than call centers ! ?).

Unpredictable Variability: The Multi-Class Case

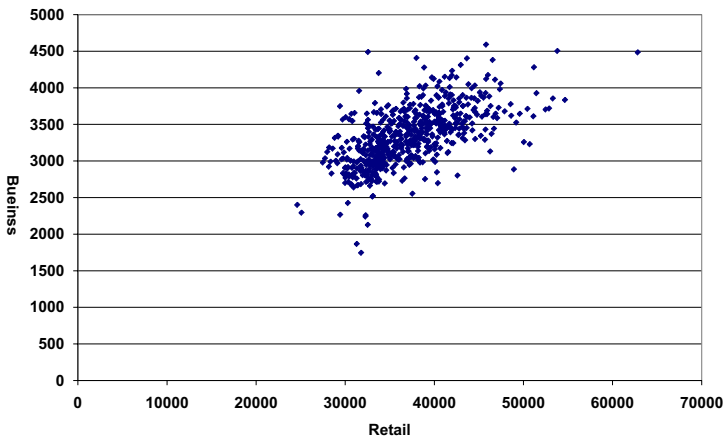
- Research w/ I. Gurvich & P. Liberman, ongoing.

Unpredictable variability: $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_I)$

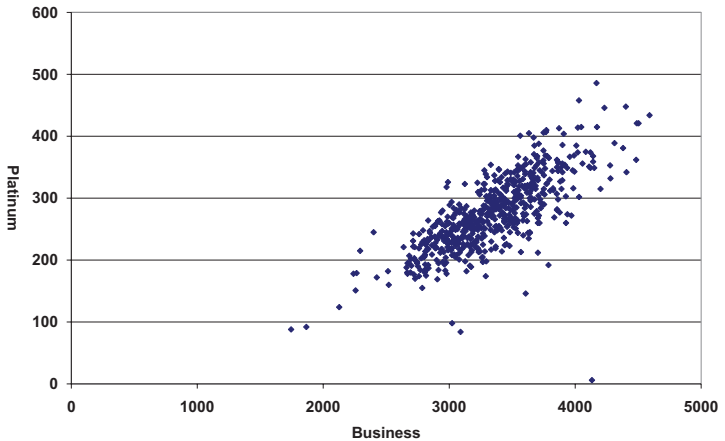
Pairs: $(\mathbf{X}_{Retail}, \mathbf{X}_{Business})$ and $(\mathbf{X}_{Business}, \mathbf{X}_{Platinum})$

US Bank: Correlations, 600 weekdays

Business vs. Retail



Business vs. Platinum



- Positive correlation (vs. independent in existing research)
- Research: Empirical, then Impact on design and control ?