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Graphical models for manpower planning

RANDOLPH W. HALL†

A work-force planning model is developed for minimizing the costs of standard wages, over-time wages, hiring and firing. The optimal work-force size and hiring/firing plan are found by plotting a labour requirement curve and calculating the change in cost due to infinitesimal changes in work-force size. Simple optimality criteria are derived by determining when the change in cost equals zero. Based on the model's assumptions, the following planning principles are identified: (1) Over-time and under-time should only be used during periods when the work-force size is not changing. (2) If work-force size is kept constant, the proportion of time in which over-time is used should equal $1/P$, where P is the ratio of the over-time wage rate to the standard wage rate. (3) Cost changes gradually when the work-force size is either made larger or smaller than the optimum. Therefore, a wide range of work-force sizes yields costs that are close to the optimum.

Introduction

Aggregate production planning and aggregate manpower planning are two of the most studied subjects in operations research. A recent textbook in production management by Hax and Candea (1984) contained well over a hundred references on this subject. And aggregate planning continues to be one of the most common subjects in operations research journals (e.g. Rahman and Nachlas 1983, Wijngaard 1983).

The aim of aggregate planning is to plan for resource acquisition and deployment in expectation of future change. It provides guidelines for when to use over-time, hire or fire employees, or subcontract, in order to meet requirements. Most importantly, the key to aggregate planning is deciding what should be done now to prepare for the future.

In the academic literature, the predominant direction in planning is reflected in the papers by Bowman (1956), Hanssmann and Hess (1960) and Hu and Prager (1959). The common approach of these papers was to divide demand requirements into discrete time periods, and model production, inventory and back-orders with a network flow model. By assuming a linear cost structure, manpower and production schedules can then be solved with linear programming, or in some cases with the transportation method of linear programming.

Another approach to aggregate planning is also based on the notion of dividing requirements into discrete time periods, and modelling production, inventory and back-orders with network flows, but assumes a quadratic cost structure instead of a linear structure (Holt *et al.* 1960). The advantage of this approach is that the resulting decision rules are simple linear functions. Much of the literature in aggregate manpower and production planning is reviewed in two excellent articles by Eilon (1975) and Silver (1967).

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† Department of Industrial Engineering and Operations Research, University of California, Berkeley, California 94720, U.S.A.

Despite the worthwhile investment in planning techniques, organizations have not been quick to implement them. Among the reasons are that computer programming requirements can be enormous and available models do not account for all the factors affecting costs. Moreover, because planning typically involves allocating expensive labour resources, managers are reluctant to relegate decisions to computer models. Over-time, hiring/firing and allocation decisions are often made at the highest level of the organization. Hence, it may be unrealistic to expect managers to turn this job over to computer programs written by analysts.

This paper attempts to develop models that provide a better understanding of planning costs and trade-offs. It is hoped that managers will make better decisions if they understand these trade-offs. To accomplish this goal, several common decisions are examined in a somewhat simplified context. The aim is to develop a *normative* model that highlights and clarifies fundamental cost trade-offs—trade-offs that are obscured by conventional planning models (e.g., linear programming). The normative model is pedagogic.

Rather than rely on algebraic expressions, this paper will focus on graphs and figures. The approach taken is similar to Newell's in his book on queueing (Newell 1982), and to unpublished notes written by Newell on production planning.

The paper begins by illustrating a simple trade-off between over-time labour costs and standard labour costs. Two cases are examined: (1) constant work-force size; and (2) work-force size that increases at a constant rate. Then hiring and firing costs are examined. A set of work-force planning rules is developed and interpreted. An example shows how these rules can be used to identify a cost minimizing work-force plan.

Trade-off between over-time and standard labour costs

Example 1. Constant work-force size

Figure 1 is a projection for unskilled labour requirements for a large organization over the next 80 months. Throughout this paper, this projection will be referred to as the 'labour requirement curve' or as the 'requirement curve'. Assume that labour requirements can be satisfied in either of two ways, by: (1) standard labour, or (2) over-time labour. Wage rates are represented by the following parameters: w = standard wage rate (\$/month) and Pw = over-time wage rate (\$/month), where P is interpreted as the premium for over-time labour.

Suppose that the organization desires to minimize its labour costs, but also would like to keep a constant work-force size over the next 80 months. Then a trade-off exists between standard labour cost and over-time labour cost that depends on the work-force size. The larger the work-force, the less over-time labour is required and the lower the over-time costs; the smaller the work-force, the lower the standard labour cost.

Let W = work-force size (workers), T = length of planning period (months), and $T(W)$ = length of time in which labour requirements exceeds the work-force size during the planning period (months).

$T(W)$ decreases from 80 months when W equals 1100 workers (the minimum labour requirement), to zero when W exceeds 3700 workers (the maximum labour requirement, see Fig. 2). Clearly, the work-force size should be somewhere between these boundaries. The best size depends on the relative costs for standard and over-time labour. If over-time rates are much larger than standard rates, W

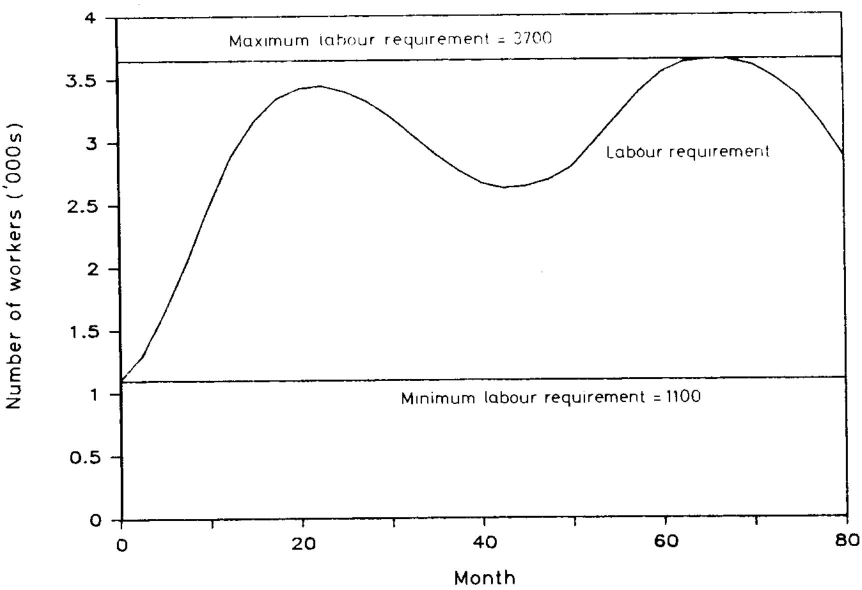


Figure. 1. Labour requirement for 80 months.

should be large; if over-time rates are nearly the same as standard rates W should be small.

The work-force size which minimizes total labour cost is easily found by examining how much total cost changes when the work-force is increased by an infinitesimal amount $d\tilde{W}$. Examining Fig. 3, the change in cost, dC, is

$$dC = (Twd\tilde{W}) - (T(W)Pwd\tilde{W}) \tag{1}$$

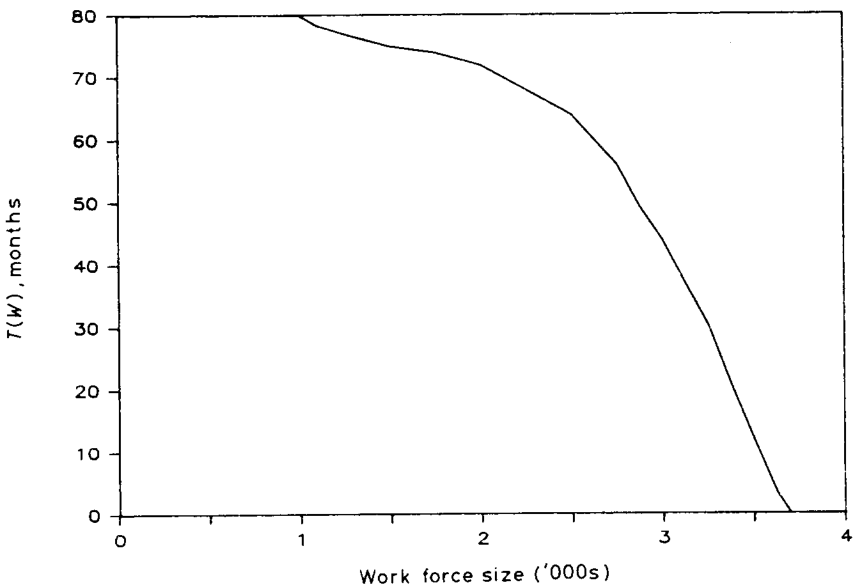


Figure 2. Months when requirement exceeds work-force versus work-force size.

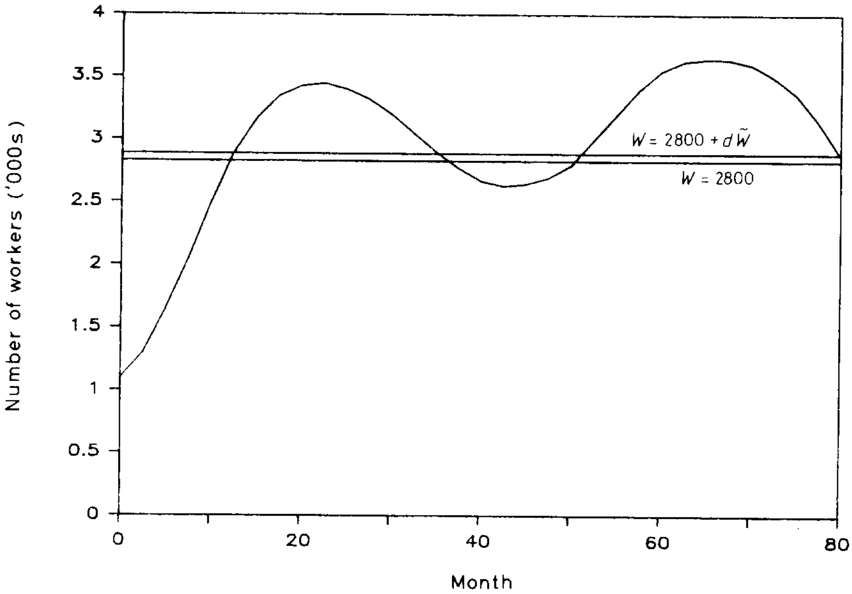


Figure 3. Cost minimizing work-force size.

The first term is the change in standard labour cost and the second term is the change in over-time labour cost. The key difference between these two terms is that over-time labour is only paid for time worked, while standard labour is paid for the entire planning period whether or not they are busy. When $T(W)$ is sufficiently large, it is less expensive to pay a worker the entire wage at standard rate than pay a partial wage at over-time rate. As W increases, the derivative in eqn. (1) increases and eventually equals zero

$$0 = Tw - T(W)(Pw) \tag{2}$$

Or, more simply

$$T/T(W) = P \tag{3}$$

Interpreting eqn. (3), if $P = 2$, the labour requirement should exceed the work-force size during 50% of the months; if $P = 1{\cdot}5$, the labour requirement should exceed the work-force size during 67% of the months.

$T/T(W)$ is a monotonically increasing function of W, varying from one to infinity as W increases from zero. Because P must be greater than 1 (over-time wages are larger than standard wages), eqn. (3) will always be satisfied at exactly one point. Hence, it is both a necessary and sufficient condition for optimality.

In Fig. 3, the optimum work-force size equals 2800, and $T(2800) = 53$ months. Hence, this work-force size minimizes cost when $P = 1{\cdot}5$.

Equation (1) can be integrated to obtain total labour cost, C, as a function of work-force size

$$C = K + \int_0^W (T - T(\tilde{W})P)wd\tilde{W} \tag{4}$$

where

$$K = \int_0^\infty T(\tilde{W})Pwd\tilde{W} \tag{5}$$

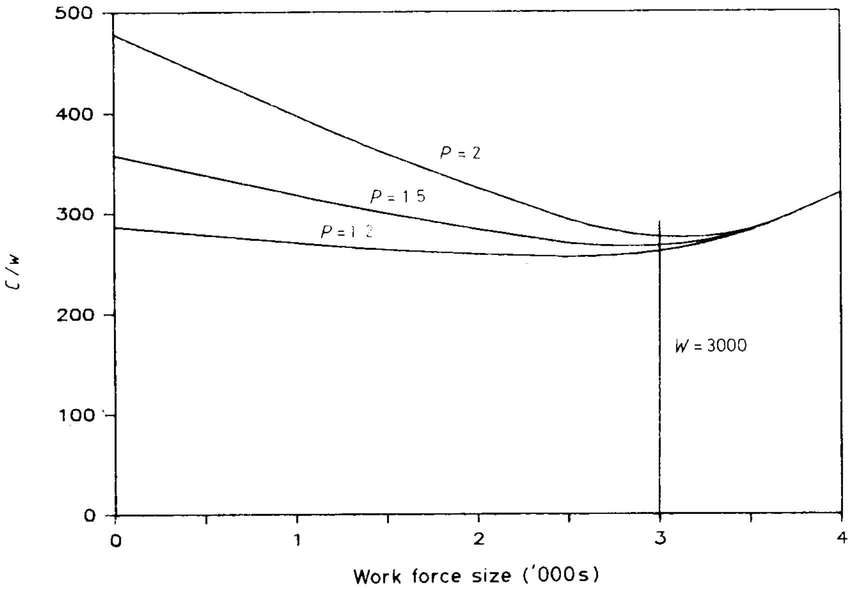


Figure 4. Cost versus work-force size.

which is to say, K is the cost of satisfying all labour requirements with over-time labour. Equations (4) and (5) can be simplified to

$$C = TWw + \int_W^\infty T(\tilde{W})Pwd\tilde{W} \tag{6}$$

or, rewritten as

$$C/w = TW + \int_W^\infty T(\tilde{W})Pd\tilde{W} \tag{7}$$

Equations (6) and (7) can be evaluated with numerical integration, either with a computer or by hand. It can also be evaluated by plotting the cumulative demand curve on a piece of paper, and counting the number of squares corresponding to standard labour, and the number of squares corresponding to over-time labour.

Figure 4 plots eqn. (7) for various values of P. The vertical axis can be interpreted as the number of 'standard labour units' used during the planning period. That is, C/w equals the number of labour months worked at the standard rate, plus P multiplied by the number of labour months worked at the over-time rate.

Figure 4 provides quite a bit of useful information for a work-force planner. For any standard wage rate, Fig. 4 shows how cost depends on the over-time premium (P). For this example, it shows that P has a small impact on the *optimal* cost. The optimal value of C/w is close to 290 for any value of P between 1·2 and 2. This information might be useful in negotiating wage rates with employees. It shows that reducing the standard wage rate has a much larger impact on cost than reducing the over-time wage rate.

Figure 4 also shows that the optimal work-force size varies from about 2500 when $P = 1{\cdot}2$, to about 3200 when $P = 2{\cdot}0$. However, it also shows that cost does not change substantially when the work-force size is not set at the precise optimum. For instance, if the work-force size equals 3000, cost will be within 3% of the optimum if P is anywhere between 1·2 and 2. Hence, if the over-time premium is expected to vary between 1·2 and 2 during the next 80 months, a work-force size of 3000 will always keep cost close to the optimum.

Even if the over-time wage rate does not vary, a work-force planner may be uncertain as to the impact of over-time labour on employee morale and productivity. Therefore, the 'true' over-time premium may be somewhat larger than wage differential. If the planner believes the true premium is 1·5, but also thinks it could be anywhere from 1·2 to 2·0, Fig. 4 suggests that a work-force size of 3000 will be very close to optimal for any value of P falling in this range.

Example 2. Hiring rate

Suppose now that the work-force size will not be constant over the planning period. Instead, employees will be hired at a more or less constant *net* rate, R, beginning from an initial work-force size of W_0. R represents the rate at which work-force size changes (which equals the hiring rate minus the firing rate). If R is greater than zero, more employees are being hired than fired. If R is less than zero, more employees are being fired than hired.

Similar to Example 1, the work-force plan is easily identified by considering how much cost changes if the hiring rate is increased by an infinitesimal amount, $d\tilde{R}$. Figure 5 shows work-force size as a function of time, for a hiring rate of $R = 225$ per year, and an initial work-force size of $W_0 = 2000$. Notice that the work-force curve crosses the labour requirement curve at more than one point. Let t_1 be the time when the curves first cross, t_2 be the time when the curves

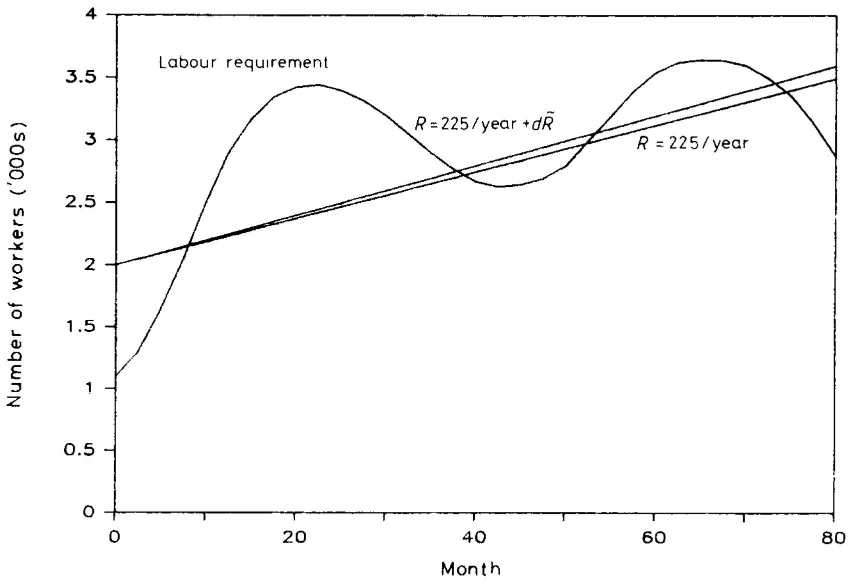


Figure 5. Cost minimizing hiring rate.

second cross, and so on. Initially, for this example, the work-force size is larger than the requirement. Therefore, the change in cost, due to a change in R, is

$$dC = 0{\cdot}5T^2wd\tilde{R} - \sum_{i=1}^{I}\left(\frac{(t_{2i}+t_{2i-1})}{2}\,d\tilde{R}(t_{2i}-t_{2i-1})wP\right) \tag{8}$$

where I equals the number of times the work-force curve crosses the requirement curve, divided by two. The summation represents over-time cost, which corresponds to a sum of the areas for a series of trapezoids. Equation (8) can be simplified to

$$dC = 0{\cdot}5wd\tilde{R}\left(T^2 - P\sum_{i=1}^{I}(t_{2i}^2 - t_{2i-1}^2)\right) \tag{9}$$

Let the term $S(R)$ denote the summation in eqn. (9). Then dC equals zero when

$$0 = 0{\cdot}5w(T^2 - PS(R)) \tag{10}$$

Equation (10) simplifies to

$$S(R) = T^2/P \tag{11}$$

$S(R)$ decreases monotonically from T^2, when R equals $-\infty$, to zero when R equals $+\infty$. Because P must be greater than one, $S(R)$ must equal T^2/P at exactly one point. Therefore, eqn. (10) is both a necessary and sufficient condition for optimality.

In Fig. 5, $S(R)$ equals 4270 months^2, and T^2 equals 6400 months^2. Hence, the work-force curve minimizes labour cost when $P = 1{\cdot}5$.

Hiring and firing costs

The situations considered so far do not explicitly consider hiring and firing costs. These costs are important for firms that vary work-force size in response to changes in labour requirement. For example, some firms hire extra employees when demand is large, and fire employees when demand is small. This strategy is effective when hiring and firing costs are small relative to over-time costs.

Assume that employees can be hired and fired at any rate. Let

- h cost of hiring an employee (\$/employee)
- f cost of firing an employee (\$/employee)
- $L(t)$ labour requirement at time t (employees)
- $R(t)$ net hiring rate at time t (employees/week)
- $W(t)$ work-force size at time t (employees)

If $R(t)$ is positive, more employees are being hired than fired at time t; if $R(t)$ is negative, more employees are being fired than hired at time t.

Hiring principle

At any time t: (1) $R(t) = 0$, or (2) $R(t) = dL(t)/dt$ and $W(t) = L(t)$.

The hiring principle states that either work-force size should be constant, or employees should be hired or fired at the same rate at which the labour requirement changes (either positive or negative). Recall that hiring and firing costs do not depend on the rates at which employees are hired and fired: they only depend on the total numbers of employees hired and fired. Hiring employees at any rate other than zero or $dL(t)/dt$ always results in larger labour costs and no change in hiring and firing costs.

Definition. $W(t)$ will be referred to as the 'work-force curve' hereafter.

The hiring principle is illustrated with six examples in Fig. 6–8. In Fig. 6 (*a*), the labour requirement increases, then decreases. At first, the work-force curve

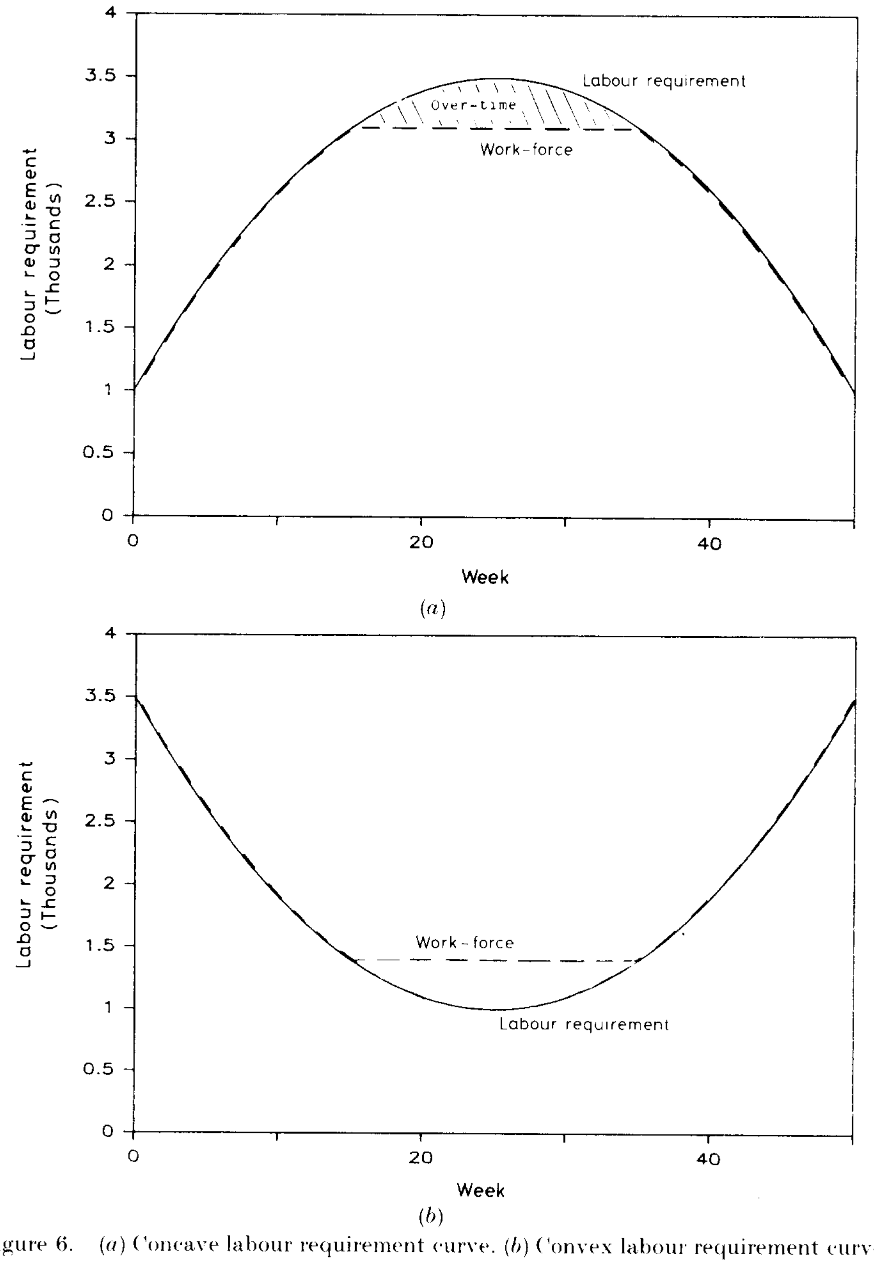


Figure 6. (*a*) Concave labour requirement curve. (*b*) Convex labour requirement curve.

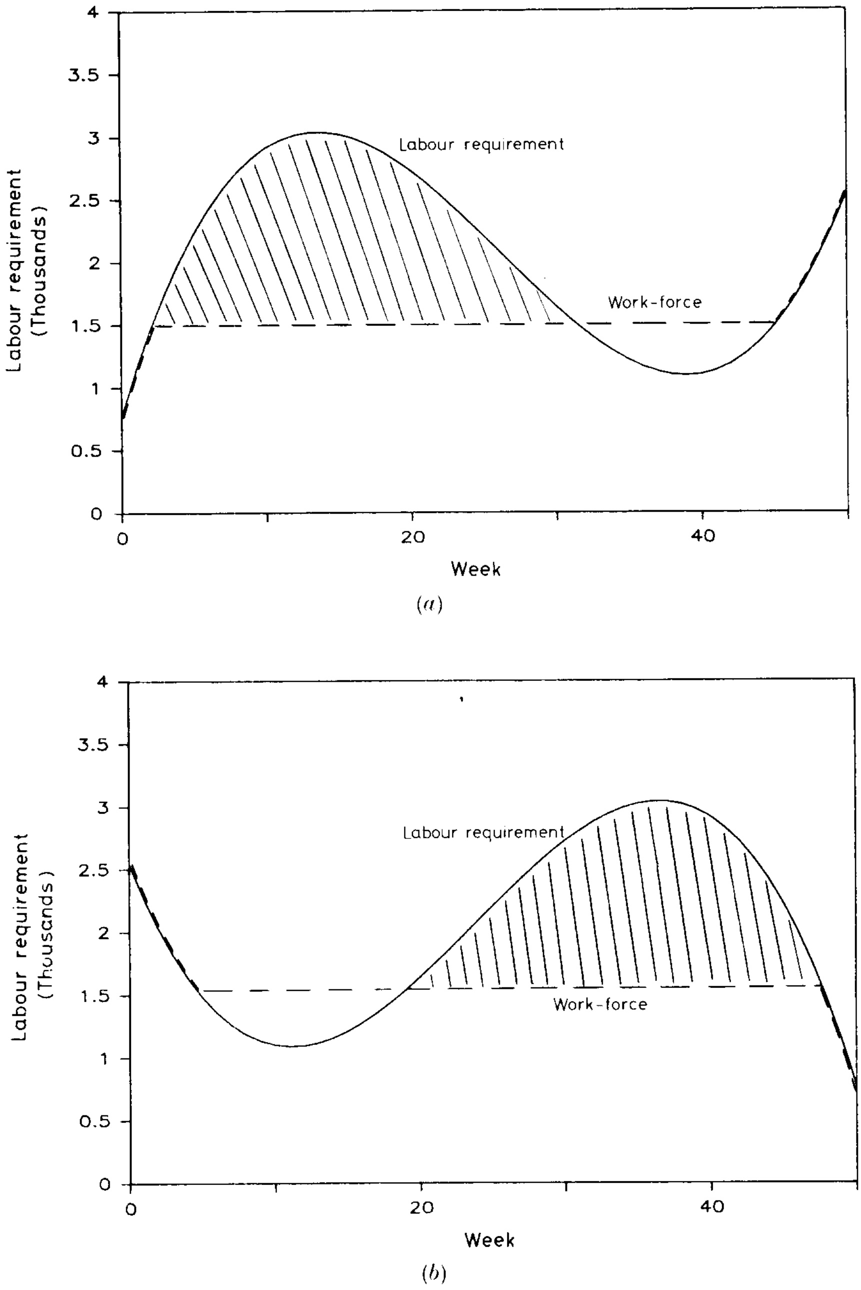
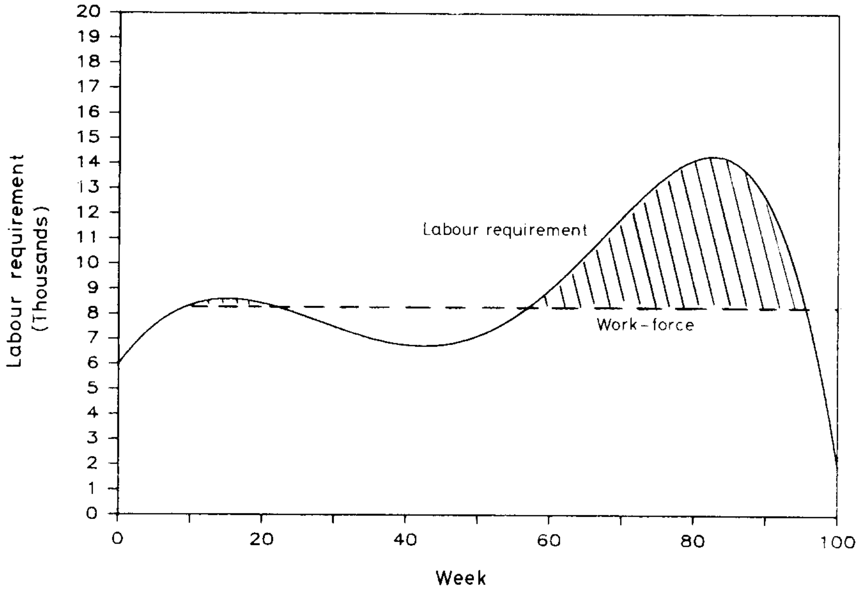


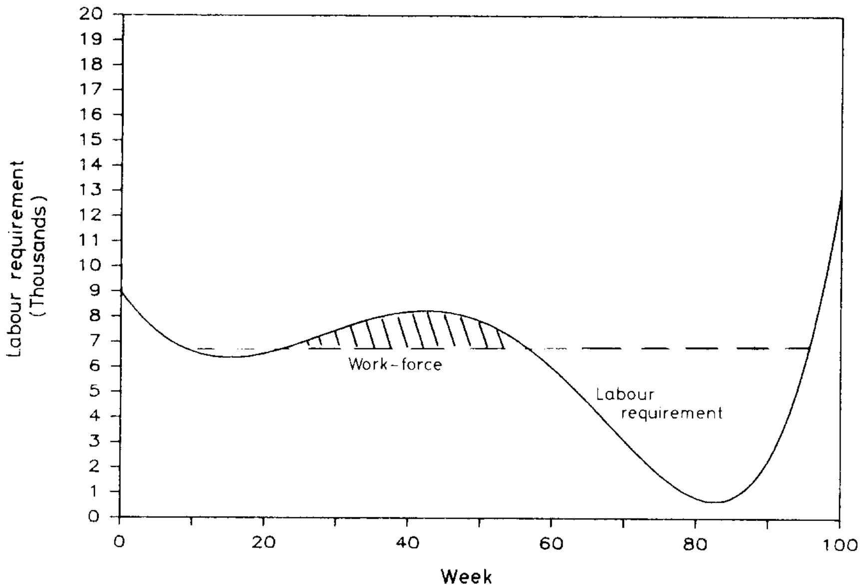
Figure 7. (*a*) Increasing work-force size. (*b*) Decreasing work-force size.

exactly matches the requirement curve, and employees are hired at the same rate that the labour requirement increases. However, after 15 weeks, the work-force size becomes constant for the next 20 weeks.

Definition. A *Constant Segment* is a period of time in which work-force size is constant.



(a)



(b)

Figure 8. (a) Increasing—constant—decreasing work-force size. (b) Decreasing—constant—increasing work-force size.

The work-force size again matches the labour requirement for the remaining weeks, and employees are fired at the same rate at which the labour requirement decreases.

Figure 6 (b) is an upside-down version of Fig. 6 (a). Employees are first fired, then work-force size becomes constant, and finally employees are hired at the same rate that the labour requirement increases.

Figures 7 (*a*) and 7 (*b*) are slightly more complicated. In Fig. 7 (*a*), the requirement curve first increases, then decreases, then increases again. Work-force size first matches the labour requirement, then becomes constant, and finally matches the labour requirement again. No employees are fired. Figure 7 (*b*) is the mirror image of Fig. 7 (*a*). Employees are fired and no employees are hired.

Figures 8 (*a*) and 8 (*b*) are more complicated. The key feature of Fig. 8 (*a*) is that employees are first hired, work-force size becomes constant, and finally employees are fired. The key feature of Fig. 8 (*b*) is that employees are first fired, work-force becomes constant, and finally employees are hired.

By now, a pattern should have emerged. (1) The hiring pattern consists of alternating periods of no hiring or firing, and periods where employees are hired and fired at the same rate that the labour requirement changes. (2) Over-time is only used during periods in which no employees are hired or fired. Moreover, the optimal work-force curve can be identified by finding the best constant segments. Between these segments, the work-force curve exactly matches the requirement curve.

Throughout the remainder of this section, the curves depicted in Figs. 6–8 will be classified into three types, as to whether the work-force size

(1) Increases—stays constant—then decreases
(2) Decreases—stays constant—then increases;
(3) Increases—stays constant—then increases, or
decreases—stays constant—then decreases.

Type 1. Special case: The work-force curve in Fig. 6 (*a*) is a special case of Type 1. Let W be the work-force size during the constant segment, and let $T(W)$ be the length of time in which $L(t) > W(t)$ (which, in this case, also equals the length of the constant segment). For each additional employee hired during the constant segment, the firm incurs the cost of $h+f$ for hiring and firing, and saves the cost $T(W)Pw$ for overtime. Therefore, if W is increased by an infinitesimal amount $d\tilde{W}$, cost would change by the following amount

$$dC = d\tilde{W}(h+f) - d\tilde{W}(T(W)Pw) \tag{12}$$

If the decrease in over-time cost is larger than the increase in hiring-firing cost, more employees should be hired.

As a necessary condition for optimality, the change in cost should equal zero

$$\begin{aligned} 0 &= d\tilde{W}(h+f) - d\tilde{W}(T(W)(Pw - w)) \\ &= (h+f) - wT(W)(P-1). \end{aligned} \tag{13}$$

Equation (13) reduces to

$$T(W) = (h+f)/w(P-1) \tag{14}$$

For example, if it costs \$6000 to hire an employee and \$4000 to fire an employee, employees are paid at the standard rate of \$2170 per month and double-time for over-time, over-time should be used for 4·6 months (20 weeks). The constant segment should also last 4·6 months.

When *hiring and firing costs* are large, over-time should be used for a longer period of time (e.g. if $h = f = \$12\,000$, and $Pw = \$2170$ per month, $T(W)$ should equal 9·2 months). This also means that the constant segment should be longer.

When the *over-time wage rate* is large, the work-force size should be kept constant for a shorter period of time, and less over-time should be used.

For unskilled labour, hiring and firing costs tend to be small relative to wage rates. This means that firms should use less over-time, and hire and fire more employees. In essence, it is more efficient for unskilled employees to change jobs between different firms (depending on who needs more workers at the moment) than to work over-time.

On the other hand, for highly skilled trades, hiring and firing costs may be very large relative to over-time wage rates. This means that firms should use more over-time and hire and fire fewer employees. In essence, it is more efficient for a firm to retain a more constant work-force size, and incur some extra over-time cost, than incur the large hiring and firing costs.

Equation (14) is also a sufficient condition for optimality. Increasing W increases hiring/firing costs at a constant rate and decreases over-time costs at a decreasing rate. As a function of work-force size, cost is a convex function. Hence, it has a unique minimum.

Type 2. Special case: Now, examine Fig. 6 (*b*). Again, let W be the work-force size over the constant segment. Also let $\bar{T}(W)$ be the length of the constant segment (months). In this case, $T(W) = 0$, which does not equal $\bar{T}(W)$. This means that $\bar{T}(W)$ equals the length of time in which some employees are idle ($W(t) > L(t)$). If W is increased by one employee, hiring or firing cost will now *decrease* by $h + f$, but standard time labour cost will *increase* by $\bar{T}(W)w$. In essence, increasing the number of employees means that the firm will have to pay for more idle labour hours.

If the work-force size is increased by an infinitesimal amount $d\tilde{W}$, cost will change by the following amount

$$dC = d\tilde{W}(\bar{T}(W)w - (h + f)) \tag{15}$$

The optimality condition is similar to eqn. (14)

$$\bar{T}(W) = (h + f)/w \tag{16}$$

Equation (16) shows that the length of time in which some employees are idle equals the ratio of the hiring and firing cost to the *standard* wage-rate. When this ratio is large, employees should be idle for a longer period of time. The work-force curve in Fig. 6 (*b*) is optimal when $(h + f)/w$ equals 4·6 months (20 weeks).

Type 3: In Figs. 7 (*a*) and 7 (*b*), increasing W does not result in increased hiring and firing cost. It merely results in hiring or firing employees sooner (or later).

The optimal value of W can be identified in much the same way that the optimal constant work-force size was identified in the first section. Increasing W increases standard-time labour cost and decreases over-time labour cost. Again let $T(W)$ be the length of time in which $L(t) > W(t)$ and let $\bar{T}(W)$ be the length of the constant segment (which is larger than $T(W)$). Then increasing the work-force size by an infinitesimal amount $d\tilde{W}$ results in the following change in cost

$$dC = d\tilde{W}(\bar{T}(W)w - T(W)Pw) \tag{17}$$

Which leads to the following *optimality condition: Type 3*

$$\bar{T}(W)/T(W) = P \tag{18}$$

Equation (18) applies to any situation where work-force size decreases, stays constant, then resumes decreasing; or where work-force size increases, stays constant, then resumes increasing. It can be interpreted in the same way as eqn. (3). The work-force curves in Figs. 7 (*a*) and 7 (*b*) are optimal when $P = 1{\cdot}5$.

Type 1. General: Figure 8 (*a*) illustrates a general version of Figure 6 (*a*). Work-force size increases, is held constant, then decreases. However, unlike Fig. 6 (*a*), the requirement curve falls below the work-force curve during part of the constant segment. If W is increased, hiring and firing cost increases, standard labour cost increases, and over-time labour cost decreases.

If the work-force size is increased by an infinitesimal amount $d\tilde{W}$, cost will change by the following amount

$$dC = d\tilde{W}(h + f) + d\tilde{W}(\bar{T}(W)w) - d\tilde{W}(T(W)Pw) \tag{19}$$

Setting eqn. (19) to zero yields

$$0 = (h + f) + \bar{T}(W)w - T(W)Pw \tag{20}$$

which can be reduced to the following *optimality condition: Type 1*

$$(h + f)/w = T(W)P - \bar{T}(W) \tag{21}$$

In some situations, eqn. (21) might be satisfied at more than one point. It may then be necessary to compare these points to identify the global optimum value of W. The work-force curve in Fig. 8 (*a*) is optimal when $(h + f)/w = 2{\cdot}3$ months, and $P = 2$.

Notice that eqn. (21) is a more general version of eqn. (14). In Fig. 6 (*a*), $\bar{T}(W) = T(W)$. Therefore, the right side of eqn. (21) can be written as $T(W)(P - 1)$. Dividing both sides by $(P - 1)$ results in eqn. (14).

Type 2. General: A final general case is illustrated in Fig. 8 (*b*): work-force size first decreases, stays constant, then increases. Increasing the work-force size during the constant segment by an infinitesimal amount $d\tilde{W}$ changes cost by the following amount

$$dC = -d\tilde{W}((h + f) + \bar{T}(W)w - T(W)Pw) \tag{22}$$

which can be reduced to the following *optimality condition: Type 2*

$$(h + f)/w = \bar{T}(W) - T(W)P \tag{23}$$

Equation (23) is identical to eqn. (21), except that the right side has been multiplied by -1. The work-force curve in Fig. 8 (*b*) is optimal when $(h + f)/w$ equals $2{\cdot}3$ months, and $P = 2{\cdot}0$.

Notice that eqn. (23) is a more general version of eqn. (18). In Fig. 6 (*b*), $T(W) = 0$. Therefore, the right side of eqn. (23) becomes $\bar{T}(W)$, and eqn. (23) becomes identical to eqn. (18).

General labour requirement pattern

The optimality conditions derived in the previous section are the building blocks for developing more complicated work-force plans. With a small amount of effort, a planner can easily arrive at a cost minimizing work-force plan, without using a computer.

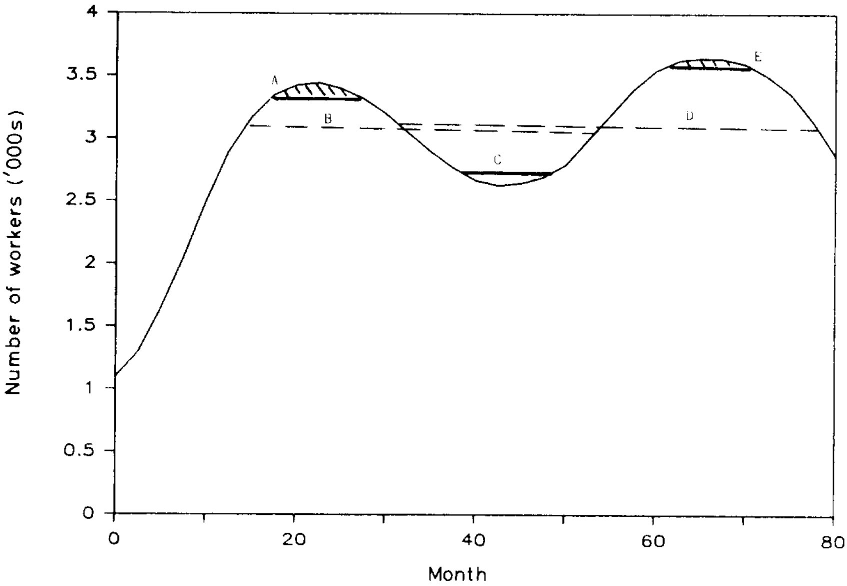


Figure 9. Work-force plan with hiring and firing.

To develop the work-force plan, it is sufficient to identify the best constant segments. At all other times, the work-force curve should exactly match the requirement curve. Second, each constant segment must satisfy *one* of the optimality conditions identified in the previous section (either eqn. (21) for Type 1; eqn. (23) for Type 2; or eqn. (18) for Type 3).

Figure 9 repeats the labour requirement pattern shown originally in Fig. 1 along with all the possible constant segments (these are easily identified with a simple search process). Finding the best plan amounts to no more than selecting the best combination of these constant segments. Without loss of generality, this plan will be derived under the assumption that work-force size equals labour requirement at the beginning and end of the planning period.

In many situations, the number of feasible combinations is small, and it is not difficult to enumerate and compare the options (for any given initial work-force size, there are just three possibilities in Fig. 9). However, because many of the constant segments are mutually exclusive, some can be eliminated on inspection. For example, consider segments A, B and C. A is a Type 1 segment and is above B (a Type 3 segment), which is above C (a Type 2 segment).

Clearly, the combination of A and C is better than B. According to the optimality condition for Type 1 segments, increasing the work-force size in the peak up to A must reduce cost; and according to the optimality condition for Type 2 segments, reducing work-force size in the valley down to C must also reduce cost. Therefore, segment B must be inferior to the combination of A and C. In the same fashion, segment D is inferior to the combination of C and E. The only segments that have not been ruled out are A, C and E, which define the cost minimizing work-force plan.

Had segment A been *below* segment B, it would be easy to show that segment C must be *above* segment B. This leads to what might be called an 'illogical match': work-force is smaller in the peak than it is in the valley. Let b_1 be the length of time during segment B in which $L(t) > W(t)$ and b_2 be the length of time during segment B in which $W(t) > L(t)$. By eqn. (18)

$$b_1 = b_2/(P - 1) \tag{24}$$

By eqn. (14), segment A must have the following length

$$a = (h + f)/w(P - 1) \tag{25}$$

which, by assumption, is greater than b_1. This implies that

$$\begin{aligned} (h + f)/w(P - 1) &> b_2/(P - 1) \\ (h + f)/w &> b_2 \end{aligned} \tag{26}$$

By eqn. (16), the left side of eqn. (26) must equal the length of C. Hence, the fact that segment A is below segment B implies that segment C must be above segment B (and vice-versa). An 'illogical match' such as A–C must always be worse than segment B. Hence, A and C could be eliminated from consideration.

Summary

This paper has developed a graphical model for evaluating work-force plans. The model was interpreted by considering how much cost changes when the work-force size is increased or decreased by an infinitesimal amount. In the first section of the paper, the model was used to analyse the trade-off between standard and over-time labour cost for a constant work-force size. It was then used to analyse the same trade-off for a work-force that increases at a constant rate. In the second section, the model was used to analyse the trade-off between hiring and firing costs, standard labour cost, and over-time labour cost, for a work-force that is allowed to increase and decrease over the planning period.

A strength of graphical models is their ability to highlight the solution and important cost trade-offs. A quick glance at the work-force and labour-requirement curves shows how large the work-force should be, when employees should be hired and fired, when over-time is needed, and when employees are idle. Sensitivity analyses show how much cost changes if the work-force is altered or if parameters are mis-estimated.

Even when more complicated techniques (such as linear programming) are used, graphical models are invaluable in explaining and interpreting the solution. A simple plot can sometimes convert a page of numbers into a revealing picture. In this way, graphical models facilitate understanding—the kind of understanding that leads to better decisions.

Un modèle de planification de la main d'œuvre est développé pour minimiser les coûts des salaires de base, des heures supplémentaires, d'embauche et de débauche. Le volume de la main d'œuvre optimale et les plans d'embauche/débauche sont déterminés en traçant une courbe des besoins en main d'œuvre et en calculant les changements infinitésimaux du volume de la main d'œuvre. On en tire des critères d'optimalité simples en déterminant le point où le changement de coût égale zéro. Fondés sur les suppositions du modèle, les principes de planification suivants sont identifiés: (1) Les heures supplémentaires et les périodes de non travail à plein temps ne devraient être utilisées que durant les périodes où le volume de la main d'œuvre ne change pas. (2) Si

le volume de la main d'œuvre reste constant, la proportion du temps pendant lequel on a recours aux heures supplémentaires devrait égaler $1/P$, P étant le rapport du taux de salaire des heures supplémentaires avec le taux de salaire de base. (3) Les coûts changent progressivement lorsque le volume de la main d'œuvre est supérieur ou inférieur au maximum. Par conséquent, une grande variation des volumes de la main d'œuvre entraîne des coûts qui frisent le maximum.

Es wird ein Modell zur Planung der Belegschaft entwickelt, das die Kosten der Normallöhne, Überstundenlöhne, Anstellung und Entlassung minimiert. Die optimale Größe der Belegschaft und der beste Anstell-/Entlassungsplan wird durch Auftragen einer Arbeiter-Bedarfskurve und Berechnung der Kostenänderung bei infinitesimalen Änderungen in der Größe der Belegschaft ermittelt. Einfache Optimalitätskriterien werden dadurch abgeleitet, daß der Punkt ermittelt wird, an dem die Kostenänderung gleich Null ist. Auf Grund der Annahmen, die dem Modell zugrunde liegen, ergeben sich die folgenden Planungsregeln: (1) Von Über- oder Unterstunden soll nur in solchen Perioden Gebrauch gemacht werden, in denen die Größe der Belegschaft konstant bleibt. (2) Bei konstanter Belegschaftsgröße sollte der Überstundenanteil gleich $1/P$ sein, wobei P das Verhältnis des Überstundentarifs zum Normallohn-Stundensatz bedeutet. (3) Wenn die Belegschaftsgröße geändert wird, so daß sie über oder unter dem Optimimum liegt, ändern sich die Lohnkosten nur allmählich. Infolgedessen liegen die Löhne eines breiten Belegschaftsgrößenbereichs nahe beim Optimum.

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