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Erlang B/C/A in the **QED Regime**

0. **Erlang** (1924/1913): "On the *rational* determination of the number of circuits."

   - Published in "The Life and Works of A.K. Erlang", by Brockmeyer, Halstrom and Jensen, pg. 216–221, 1948.
   - Says the formula was used in "our company" for a long time, "although without any absolute satisfactory statement of reasons".
   - Covers Erlang-B and Erlang-C, and in both $\boxed{x = y + h\sqrt{y}}$.
   - Supports the arguments via Tables.
   - Proofs are given on pg. 120 of the book: "Appendix 2: Approximative Formulae for Loss and 'Improvement' ".

Review:

1. Erlang-B (M/M/N/N, in fact M/GI/N/N): Jagerman, 1974.
2. Erlang-C (M/M/N, in fact GI/M/N): Halfin-Whitt, 1981.
3. $M_t/M/N_t$ (formalizing Jennings, et al.): Massey, Rider; ongoing.

4. Erlang-A (M/M/N + M): Fleming, Stolyar, Simon 1994; Garnett, Reiman, 1998.
5. Erlang-A with General Patience (M/M/N + GI): Zeltyn, ongoing.

6. GI/PH/N: only process limits, by Puhalskii and Reiman, 2000.
7. Simulation experiments, by Roy Schwartz, 2002 (in web-site): Note M/LN/N.
8. GI/D/N: with Jelenkovic & Momcilovic, in website.
9. Recent articles by Whitt (see his web-site).
10. SBR: Borst & Seri; ∞-server heuristics (Armony, Gurvich).

Erlang-B (M/M/N/N) Jagerman, 1974

As $N \uparrow \infty$, $\sqrt{N}$ P(Blocked) $= \sqrt{N} E_{1,N} \to \alpha$, $0 < \alpha < 1$

iff $\sqrt{N}(1-\rho_N) \to \beta$, $-\infty < \beta < \infty$, $\left(\rho_N = \frac{\lambda_N}{\mu N} = \frac{R_N}{N}\right)$,

(equivalently $N \approx R_N + \beta\sqrt{R_N}$)

in which case $\alpha = h(-\beta)$ (hazard rate of $N(0,1)$)

Erlang-C (M/M/N) Halfin & Whitt, 1981

As $N \uparrow \infty$, P(Wait > 0) $= E_{2,N} \to \alpha$, $0 < \alpha < 1$ ← customer

iff $\sqrt{N}(1-\rho_N) \to \beta$, $0 < \beta < \infty$, ← server

(equivalently $N \approx R_N + \beta\sqrt{R_N}$) ← manager

in which case $\alpha = [1 + \beta/h(-\beta)]^{-1}$, the Halfin-Whitt function.

Erlang-A (M/M/N + M) with Garnett & Reiman, 2002 [1]

As $N \uparrow \infty$, P(Wait > 0) $\to \alpha$, $0 < \alpha < 1$

iff $\sqrt{N}(1-\rho_N) \to \beta$, $-\infty < \beta < \infty$ (equiv: $N \approx R_N + \beta\sqrt{R_N}$)

iff $\sqrt{N}$ P(Abandon) $\to \gamma$, $0 < \gamma < \infty$,

in which case $\alpha = \left[1 + \dfrac{h(\delta)/\delta}{h(-\beta)/\beta}\right]^{-1}$, $\delta = \beta\sqrt{\mu/\theta}$

and $\gamma = \alpha\beta\left[\dfrac{h(\delta)}{\delta} - 1\right]$, (What if $\mu = \theta$?)

Erlang-A/G (M/M/N + G) with Zeltyn, 2003 [2]

The results for Erlang-A carry over as is, under the assumption that Impatience has a density which is positive at the origin. Then, denoting this density value by $g(0) > 0$, the performance measures for M/M/N + G are precisely those for M/M/N + N, with simply substituting $g(0)$ for θ.

Note: If $g(0) = 0$, or if there is balking (Patience with positive mass at the origin), the above QED characterizations must be modified.

[1] http://iew3.technion.ac.il/serveng/References/abandon02.pdf

[2] http://iew3.technion.ac.il/serveng/References/MMNG_formulae.pdf

Proof – Erlang-B:

M/G/N/N : $\quad \mathrm{P(blocked)} = \underset{\substack{\uparrow \\ \text{PASTA}}}{\pi_N} = E_{1,N}(a) = \dfrac{a^N/N!}{\sum_{k=0}^{N} a^k/k!} \quad \left(a = \frac{\lambda}{\mu}\,, \text{ denoted previously by } R\right)$

$$E_{1,N} = \frac{e^{-a}a^N/N!}{\sum_0^N e^{-a}a^k/k!} = \frac{P(X_a = N)}{P(X_a \le N)} \quad , \qquad X_a \sim \text{ Poisson } (a)$$

$$= \frac{P(N-1 < X_a \le N)}{P(X_a \le N)}$$

Denominator $= P\left(\dfrac{X_a - a}{\sqrt{a}} \le \dfrac{N-a}{\sqrt{a}}\right) \to P(\overset{\substack{N(0,1)\\ \downarrow}}{Z} \le \beta) = \phi(\beta)$

assuming $\boxed{\dfrac{N - a_N}{\sqrt{a_N}} \to \beta}$, $\qquad -\infty < \beta < \infty$, and using the CLT.

Numerator $= P\left(\dfrac{N-a-1}{\sqrt{a}} < \dfrac{X_a - a}{\sqrt{a}} \le \dfrac{N-a}{\sqrt{a}}\right) \approx$

$$\approx P\left(\beta - \frac{1}{\sqrt{a}} < Z \le \beta\right) = \phi(\beta) - \phi\left(\beta - \frac{1}{\sqrt{a}}\right)$$

$$\approx \frac{1}{\sqrt{a}}\,\varphi(\beta)\ , \qquad \text{under the denominatoer limits.}$$

Conclude: If $\quad \dfrac{N-a_N}{\sqrt{a_N}} \to \beta, \quad$ then $\quad \sqrt{a_N}E_{1,N}(a_N) \to \dfrac{\varphi(\beta)}{\phi(\beta)}.$

Note: $\dfrac{\varphi(\beta)}{\phi(\beta)} = \dfrac{\varphi(-\beta)}{1-\phi(-\beta)} = h(-\beta) \quad$ hazard rate.

Lemma $\quad (N - a_N)/\sqrt{a_N} \to \beta \qquad \Rightarrow \rho_N \to 1, \qquad -\infty < \beta < \infty.$

Corollary $\quad (N - a_N)/\sqrt{a_N} \to \beta \qquad \Leftrightarrow \sqrt{N}(1-\rho_N) \to \beta$

(since $\quad \sqrt{N}(1-\rho_N) = [(N-a_N)/\sqrt{a_N}] \cdot \sqrt{\rho_N} \to \beta \quad$)

Theorem (M/G/N/N) $\quad \sqrt{N}\ (1-\rho_N) \to \beta, \quad -\infty < \beta < \infty\ , \quad$ is equivalent to

$$\sqrt{N}P(Blocked) = \sqrt{N}\ E_{1,N} \to h(-\beta).$$

$\sim$

Proof – Erlang-C, simply based on Palm's relation between $E_{2,N}$ and $E_{1,N}$:

$$P(\text{Wait } > 0) = E_{2,N} = \frac{E_{1,N}}{(1-\rho) + \rho E_{1,N}} \ .$$

$$\begin{aligned} E_{2,N} &= \left[\rho_N + \frac{1-\rho_N}{E_{1,N}}\right]^{-1} = \left[\rho_N + \frac{\sqrt{N}(1-\rho_N)}{\sqrt{N}E_{1,N}}\right]^{-1} \\ &\to \left(1 + \frac{\beta}{h(-\beta)}\right)^{-1}, \text{ iff } \sqrt{N}(1-\rho_N) \to \beta\,, \quad 0 < \beta < \infty, \end{aligned}$$

which yields

Halfin-Whitt's Theorem (Erlang-C)

$$E_{2,N} \to \alpha\,, \quad 0 < \alpha < 1 \;\Leftrightarrow\; \sqrt{N}(1-\rho_N) \to \beta, \quad 0 < \beta < \infty$$

$$(\Leftrightarrow \;\; N \approx R_N + \beta\sqrt{R_N}\;)$$

in which case $\alpha(\beta) = [1 + \beta/h(-\beta)]^{-1}$.

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For all that you ever wanted to know about Erlang B/C, see Whitt's webpage:

- http://www.columbia.edu/~ww2040/questions1e.pdf, and
- http://www.columbia.edu/~ww2040/economy.pdf