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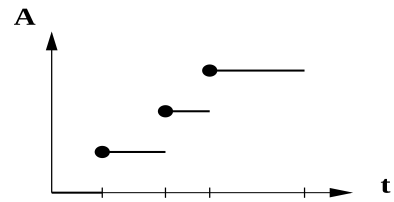
# Dynamic Randomness: The Poisson Process

Hall, Chapter 3: The *Arrival Process*

*Counting Process* $A = \{A_t,\ t \geq 0\}$, where $A_t$ = cumulative number of arrivals during $[0, t]$.

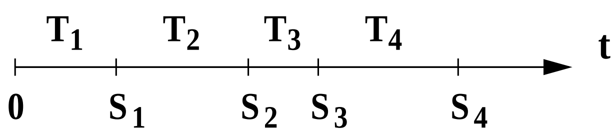
Assume: $A_0 = 0$; a single arrival at a time.

Characterization via sample paths of $A$:




or via times of *events* = jumps $S_1, S_2, S_3, \ldots$

or via *inter-arrival times* $T_1, T_2, \ldots$ : $S_n = T_1 + \cdots + T_n,\ n \geq 1$.




- Completely *deterministic* arrivals at a constant rate $\lambda$ : $T_n \equiv \frac{1}{\lambda}$ .
- Completely *random* arrivals at a constant rate $\lambda$ : ?

Today: a mathematical *model* for completely random arrivals at a *constant* rate. (Later: *varying rates.*)

**Contents**

- Mathematical Framework: Levy Processes;
- Constructions:

  Intuitive (via Bernoulli ⇒ Poisson);
  Explanatory (via "must" properties: order-statistics);
  Axiomatic (Levy + counting);
  Practical (exponential interarrivals).
- Properties; PASTA; Biased-sampling & paradoxes.
- Inference & simulation.

**Hammel's Theorem**

**Additive:** *All (measurable) solutions $f$ to the functional equation*

$$f(s+t) = f(s) + f(t), \quad \forall\ s,t \in \mathbb{R}^1,$$

*are of the form $f(t) = c \cdot t$, for some $c \in \mathbb{R}^1$.*
*(measurable $\Leftarrow$ monotone, continuous, RCLL,...)*

**Proof** $f(na) = nf(a)$, $n = 1,2,\ldots$, by induction; $(\Rightarrow mf(\frac{1}{m}) \equiv f(1))$

rationals: $f(\frac{n}{m}) = f(n \cdot \frac{1}{m}) = nf(\frac{1}{m}) = \frac{n}{m} \cdot mf(\frac{1}{m}) = \frac{n}{m}f(1)$,

continuity: $f(x) = xf(1) \quad \forall x \in \mathbb{R}^1$ (this is stronger than actually assumed). **Q.E.D.**

**Multiplicative:** *All (measurable) solutions $f$ to the functional equation*

$$g(s+t) = g(s) \cdot g(t), \quad \forall s,t \geq 0,$$

*are of the form $g(t) = e^{c \cdot t}$, $t \geq 0$, for some $c \in \mathbb{R}^1$.*

**Application** to the **Poisson Process**, say $A = \{A_t,\ t \geq 0\}$:

1. $m(t) = \mathrm{E}A_t$ : $m(t+s) = m(t) + m(s) \Rightarrow m(t) = \lambda t, \quad \lambda > 0$;

$$\lambda \equiv \mathrm{E}\left(\frac{1}{t}\ A_t\right) = \frac{1}{t}\mathrm{E}A_t \qquad \textit{arrival rate} \quad (\equiv \text{constant})$$

2. $p(t) = P\{A_t = 0\}$ : $p(t+s) = p(t)p(s) \Rightarrow p(t) = e^{-\lambda t}, \quad \lambda > 0$ ,
   $\Rightarrow$ time till the first arrival is $\exp(\lambda) \Rightarrow$ interarrival times are $\exp(\lambda)$.

3. $g(t) = E(e^{-\alpha A_t}) : g(t+s) = g(t)g(s) \Rightarrow g(t) = e^{tC(\alpha)}, \quad t \geq 0$;
   Using infinitesimal properties and $\frac{\partial}{\partial t} g(t)\Big|_{t=0} = C(\alpha) \Rightarrow C(\alpha) = -\lambda(1 - e^{-\alpha})$.

See Theorem (1.9) in Cinlar, page 74.

## Mathematical Framework: Levy Processes

Discrete-time: Random Walk

$$
\begin{aligned}
S(n) &= \Delta_1 + \cdots + \Delta_n, \quad n \geq 0, \quad \text{where } \Delta_1, \Delta_2, \ldots, \text{i.i.d. r.v.} \\
S(0) &= 0\ .
\end{aligned}
$$

Properties:

1. $S(m+n) - S(m) \overset{d}{=} S(n) - S(0) \quad \forall\, m, n \geq 0$ ($\overset{d}{=}$ same distribution)

2. $S(m_1) - S(0),\ S(m_2) - S(m_1),\ S(m_3) - S(m_2), \ldots$ independent $\forall\, m_1 \leq m_2 \leq \cdots$

$S = \{S(n),\ \ n \geq 0\}$ has **stationary** (1) and **independent** (2) increments.

The continuous-time analogue is a

**Levy process** A stochastic process $X = \{X_t,\ t \geq 0\}$ is a Levy process if

(0) $X(0) \equiv 0$ (for simplicity);

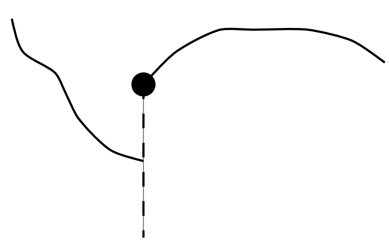
(1) $X$ has *stationary* increments, that is
$X(t+\tau) - X(t) \overset{d}{=} X(\tau) \quad \forall\, t,\ \tau \geq 0$;

(2) $X$ has *independent* increments, that is
$X(t+\tau) - X(t)$ independent of $\{X(s),\ s \leq t\}$, $\quad \forall\, t,\ \tau \geq 0$;

equivalently, $X(t_1),\ X(t_2) - X(t_1),\ X(t_3) - X(t_2) \ldots$ independent $\ \forall\, t_1 \leq t_2 \leq \cdots$

(*Technical*) (3) $X$ is continuous in probability: $\lim_{t \to 0} P\{|X_t| > \epsilon\} = 0, \ \ \forall \epsilon > 0$.

(*Convention*) (4) $X$ has sample paths that are **R**ight-**C**ontinuous with **L**eft **L**imits (RCLL).



**The Distribution of a Levy Process.** (Probabilistic Characterization.)

The *finite-dimensional distributions* are determined by marginals:

| | | | | |
|---|---|---|---|---|
| $X(t_1), X(t_2), X(t_3), \ldots \Leftrightarrow$ | $X(t_3) - X(t_2)$, | $X(t_2) - X(t_1)$, | $X(t_1) - X(0), \ldots$ | independent |
| | $X(t_3 - t_2)$ | $X(t_2 - t_1)$ | $X(t_1), \ldots$ | stationary |

In fact, they are determined by $X(1)$!

Reason: Each $X_t$ has a distribution that is *infinitely divisible*, namely

$$X(t) = \underbrace{X\left(t\,\frac{n}{n}\right) - X\left(t\cdot\frac{n-1}{n}\right)} + \underbrace{X\left(t\cdot\frac{n-1}{n}\right) - X\left(t\cdot\frac{n-2}{n}\right)} + \cdots + \underbrace{X\left(t\cdot\frac{1}{n}\right) - X(0)}\,,$$

which is the sum of $n$ i.i.d. r.v., $\forall\, n = 1, 2, 3, \ldots$

Hence, the characteristic functions $\varphi_t(u) = E(e^{iuX_t}),\ u \geq 0$, satisfy

$$\begin{aligned}
\varphi_{s+t}(u) &= Ee^{iuX_{s+t}} = Ee^{iu(X_{s+t}-X_t)}e^{iuX_t} = \text{ (independent increments)}\\
&= Ee^{iu(X_{s+t}-X_t)}Ee^{iuX_t} = \text{ (stationary increments)}\\
&= Ee^{iuX_s}Ee^{iuX_t} = \varphi_s(u)\varphi_t(u)\,,\ \forall\, s,t \geq 0\,.
\end{aligned}$$

$$\begin{aligned}
\text{Hammel } &\Rightarrow\ \varphi_t(u) = \exp[t\cdot\psi(u)] = [\varphi_1(u)]^t, \quad t \geq 0\\
&\Rightarrow\ \forall\, t \geq 0,\ \text{marginal distribution of } X_t \text{ is determined by } X_1.
\end{aligned}$$

**Fact**: There exists a complete characterization of infinitely divisible distributions ($\psi(u)$).

| | | | | |
|---|---|---|---|---|
| *Examples*: | deterministic, | Poisson, | Compound Poisson, | Normal |
| | $c$ | $\lambda$ | $\lambda, F$ | $\mu, \sigma^2$ |

**Theorem 1** *There is 1–1 correspondence between infinitely divisible distributions and Levy processes, as follows:*

- *If $X$ is Levy, then $X_1$ has infinitely divisible distributions;*
- *Conversely, given the characteristic function $\varphi(u)$ of an infinitely divisible distribution, there exists a unique Levy process $X$ whose state at time $t = 1$, $X_1$, has a characteristic function $\varphi$.*

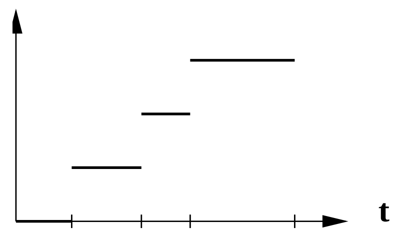
**Examples**

| | |
|---|---|
| $X_1 \equiv \mu$, then $X(t) = \mu \cdot t$, | uniform motion |
| $X_1 \stackrel{d}{=}$ Poisson $(\lambda)$, then $X(t) \stackrel{d}{=}$ Poisson $(\lambda t)$, | Poisson process |
| $X_1 \stackrel{d}{=}$ Compound Poisson, then $X$ is | Compound Poisson process |
| $X_1 \stackrel{d}{=}$ Normal $(\mu, \sigma^2)$, then $X(t) \stackrel{d}{=} N(\mu t, \sigma^2 t)$, | Brownian motion. |

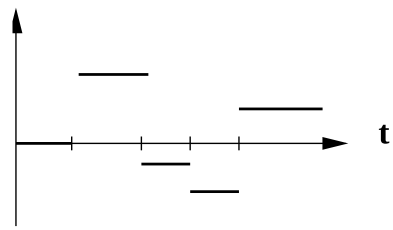
**Theorem 2** *$X, Y$ independent Levy processes $\Rightarrow X + Y = \{X_t + Y_t,\ t \geq 0\}$ is also Levy process.*

**Modeller's Dream** (from "qualitative" to "quantitative")

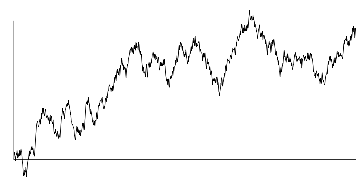
1. A Levy *counting* process is *Poisson*
   (Cinlar, pg. 71)




2. A Levy *jump* process is *Compound Poisson*
   (Cinlar pg. 91)
   changes state in jumps and jumps finitely
   in finite times.




3. A Levy *continuous* process is *Brownian Motion*
   (Breiman pg. 248)
   has continuous sample paths.



The "emergence" of the parameters:

Suppose $\exists\, m(t) = EX(t), \quad t \geq 0$. Then

$$\begin{aligned} m(s+t) &= E[X(t+s) - X(t)] + EX(t) = m(s) + m(t), \quad \forall\, s,t \geq 0 \\ \Rightarrow m(t) &= \mu \cdot t \;\text{ for some }\; \mu \;. \end{aligned}$$

Suppose $\exists\, V(t) = \text{Var } X(t),\; t \geq 0$. Then

$$V(s+t) = V(s) + V(t),\; \forall\, s,t \Rightarrow V(t) = \sigma^2 t, \quad \text{for some } \; \sigma \geq 0 \;.$$

**Final Practical Characterizations**

- *Poisson* process with parameter $\lambda$ (Poisson($\lambda$)): Levy and Counting;
  $X_t \stackrel{d}{=} \text{Poisson } (\lambda t), \quad t \geq 0$.

- *Compound Poisson*: $\quad X_t = \sum_{k=1}^{A_t} \Delta_k, \quad t \geq 0$, where

  $A = \{A_t,\, t \geq 0\}$ is Poisson ($\lambda$); $\Delta = \{\Delta_1, \Delta_2, \ldots\}$ iid (distribution F); $A$ and $\Delta$ independent.

- *Brownian* motion, with parameters $\mu, \sigma^2$ (BM$(\mu, \sigma^2)$): Levy continuous sample paths; $X_t \sim N(\mu t, \sigma^2 t),\ t \geq 0$.

  $\mu = 0,\ \sigma = 1 \Rightarrow$ *standard* BM (SBM).

  $X \stackrel{d}{=} \mathrm{BM}(\mu, \sigma^2) \Rightarrow X_t = \mu t + \sigma B_t,\ t \geq 0,$ with B = SBM.

# Hall, Chapter 3: **The Arrival Process** $N = \{N(t),\ t \geq 0\}$

§3.1 *Definition* 3.2 requires too much. As discussed, Levy + counting $\Rightarrow$
$\exists \lambda > 0 \ni N(t) - N(s) \sim$ Poisson $[\lambda(t-s)]$.

In particular,

$$\begin{aligned} P\{ \quad N(t+dt) - N(t) = 1\} &\ = \lambda dt + o(t) \\ \{ \qquad\qquad\qquad\quad = 0\} &\ = 1 - \lambda dt + o(t). \\ \{ \qquad\qquad\qquad\quad > 1\} &\ = o(t) \end{aligned}$$

§3.2 *Derivation* of the Poisson distribution from Bernoulli.

§3.3 *Properties* of the Poisson Process.

1. Poisson *marginals*; number of events in any interval is Poisson;

$$\begin{aligned} EN_t &= \lambda t\,, \text{Var } N_t = \lambda t \\ &\Rightarrow C = \frac{\sigma}{E} = \frac{\sqrt{\lambda t}}{\lambda t} = \frac{1}{\sqrt{\lambda t}} \text{ small for } t \text{ large.} \end{aligned}$$

2. *Interarrival times* which are iid exp $(\lambda)$.
   Beginning of proof: $P(T_1 \geq t) = P(N_t = 0) = e^{-\lambda t},\ t \geq 0$.
   This is a characterizing property that is practical for simulation.
   Extensions to $T_2, T_3, \ldots,$ and their independence, if rigorous, requires more than the "it should be apparent" in Hall, pg. 58.
3. *Memoryless* property: time till next event does not depend on the elapsed time since the last event.
4. $S_n = T_1 + \cdots + T_n \sim$ Gamma $(n, \lambda)$ = Erlang.
5. *Order-statistics* property: Given $N(t) = n$, the unordered event times are distributed as $n$ iid r.v., uniformly distributed on $[0, t]$.

$$\Rightarrow \text{simulation over } [0,t] : N(t) \sim \text{Poisson } (\lambda t);\ U_1, U_2, \ldots, U_{N(t)} \text{ iid } U[0,t]\ .$$

§3.4 *Goodness* of *Fit*

How well does a Poisson model fit our arrival process?

*Qualitative* assessments:

| | |
|---|---|
| Airplanes landing times at a single runway, during an hour: | no |
| Airplanes landing times at a large airport, during an hour: | plausible |
| Job candidates that arrive at their appointments during an hour: | no |
| Visits to a zoo, most of which arrive in groups, during an hour: | no |
| Arrival times at a bank ATM = **A**utomatic **T**eller **M**achine, during an hour: | plausible |

§3.5 *Quantitative Tests*

Graphical Tests:

cumulative arrivals vs. a straight line (Fig. 3.2)

paired successive interarrivals (Fig. 3.4)

exponential interarrivals
(How do you identify exp $(\cdot)$ when you see one? Use Histograms!)

§3.6 *Parameter Estimation*

Estimate $\lambda =$ arrival rate.

MLE (Max. Likelihood Estimator), given $A(t),\ t \leq T : \hat{\lambda} = \frac{A(T)}{T}$.

Confidence intervals for $\frac{1}{\lambda} : \frac{T}{A(T)} \pm z_\alpha \frac{T}{A(T)^{3/2}}$ (3.34)

Sample-size: for $(1-\alpha)$-confidence interval of width $w$, $N \geq [\frac{2z_\alpha}{w\lambda}]^2$.

Thus, for $w = \epsilon \cdot \frac{1}{\lambda}$, we need $N \geq [\frac{2z_\alpha}{\epsilon}]^2$.
(Eg.: 95%-confidence interval of width = 10% of mean, requires $N \geq [\frac{2\times1.96}{0.1}]^2 \approx$ 1500!)

## **PASTA** = **P**oisson **A**rrivals **S**ee **T**ime **A**verages (R. Wolff)

**Arrivals (Observations):**
$A = \{A(t),\ t \geq 0\}$,
$A(t)$ = number of arrivals in $[0, t]$.

**System:**
$X = \{X(t),\ t \geq 0\}$,
$X(t)$ = state at time $t$.

$$\textit{Time}\text{ average} \qquad \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T X(t)dt \quad \doteq \quad \bar{\tau}$$

$$\textit{Customer}\text{ average} \qquad \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} X(S_n-) \quad \doteq \quad \bar{c}$$

where $S_n$ = n-th arrival time.

**Fact** Assume

(i) $A$ is Poisson, and

(ii) $X$ adapted to $A$: $\forall\, t,\ X(t)$ is a function of $A(s),\ s \leq t$, hence it is independent of $A(u) - A(t),\ u \geq t$.

Then $\bar{\tau} = \bar{c}$, in the following precise sense:
If one limit exists, then the other exists as well, in which case they are equal.

**Proof** (Wolff): Based on $\sum_1^{A(t)} X(S_n-) - \lambda \int_0^t X(s)ds = \int_0^t X(s-)d[A(s) - \lambda s]$ being a martingale with mean $\equiv 0$.

**Note**:
- Hall, pg. 168–9, uses PASTA to establish Khinchine-Pollatzchek
- Counterexamples if (i) or (ii) violated; still, conditions not tight; see ASTA.

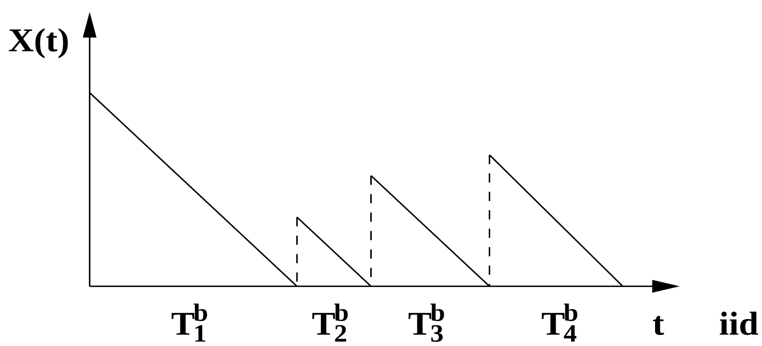
**Application** of **PASTA: Biased Sampling**

A *renewal process* is a counting process with iid interarrivals.

| | | | |
|---|---|---|---|
| Descriptions: | $R = \{R(t),\ t \geq 0\}$ or | $\{T_1, T_2, \ldots\}$ iid, or | $\{S_1, S_2, \ldots\}$ |
| Example: | Poisson | exponential | Erlang |

*Story*: Buses arrive to a bus stop according to a renewal process $R_b = \{R_b(t), t \geq 0\}$.
$T_i^b$ — times between arrivals of the buses.
Passengers arrive to the bus stop in a completely random fashion (Poisson).
$S_i^p$ — arrival times of the passengers.

*Question*: How long, on average, do they wait? Plan service-level.




$A = \{A(t),\ t \geq 0\}$ = Poisson arrivals of passengers.

$X = \{X(t),\ t \geq 0\}$ = state = *Virtual waiting time.*

PASTA: $\lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} X(S_n^p -) = \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T X(t)dt = \bar{\tau}$

$$
\begin{aligned}
\Rightarrow \bar{\tau} &= \frac{1}{T} \cdot \text{ (area under } X\text{, over } [0,T]) \\
&\approx \frac{1}{T} \cdot \left( \frac{1}{2}(T_1^b)^2 + \frac{1}{2}(T_2^b)^2 + \cdots + \frac{1}{2}(T^b_{R_b(T)})^2 \right) \\
&= \frac{R_b(T)}{T} \cdot \frac{1}{2} \cdot \frac{T_1^2 + \cdots + T^2_{R_b(T)}}{R_b(T)} \xrightarrow[T\uparrow\infty]{} \frac{1}{E(T_1^b)} \cdot \frac{1}{2} \cdot E(T_1^b)^2 \text{, by SLLN} \\
&= \underbrace{\frac{1}{2}E(T_1^b)}_{\text{“Deterministic” answer}} \qquad \underbrace{[1 + c^2(T_1^b)]}_{\text{Bias, due to variability}}, \qquad c = \frac{\sigma}{E} \text{ coefficient of variation.}
\end{aligned}
$$

Check Poisson bus arrivals to derive Paradox:
1(“stochastic” answer) = $\frac{1}{2}$ (“deterministic” answer).