Experiencing Statistical Regularity

IEOR 4106

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Experience Statistical Regularity

By Looking at Random Walks

DTMC: discrete time & discrete state

CTMC: continuous time & discrete state

Random Walks: discrete time & continuous state

Many steps: continuous time & continuous state

Simulation Experiments

Plotting Random Walks

$X_1, X_2, \ldots$ **IID random variables**

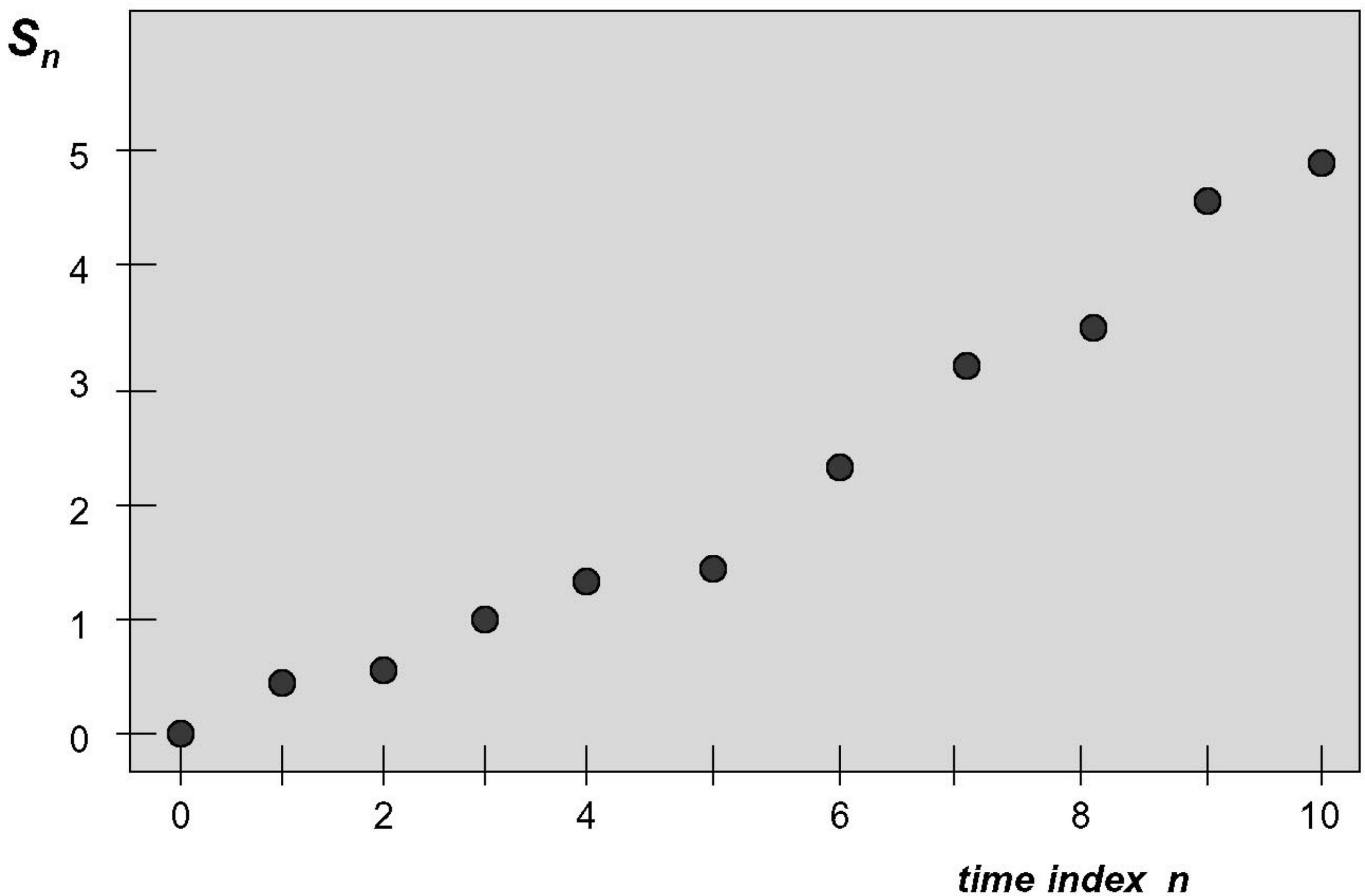
$S_n = X_1 + \cdots + X_n, \quad n \geq 1,$

with $S_0 = 0$ **partial sums**

Plot $S_0, S_1, \ldots, S_n$

To start: $X_i = U_i$ **uniformly distributed** on $[0, 1]$.

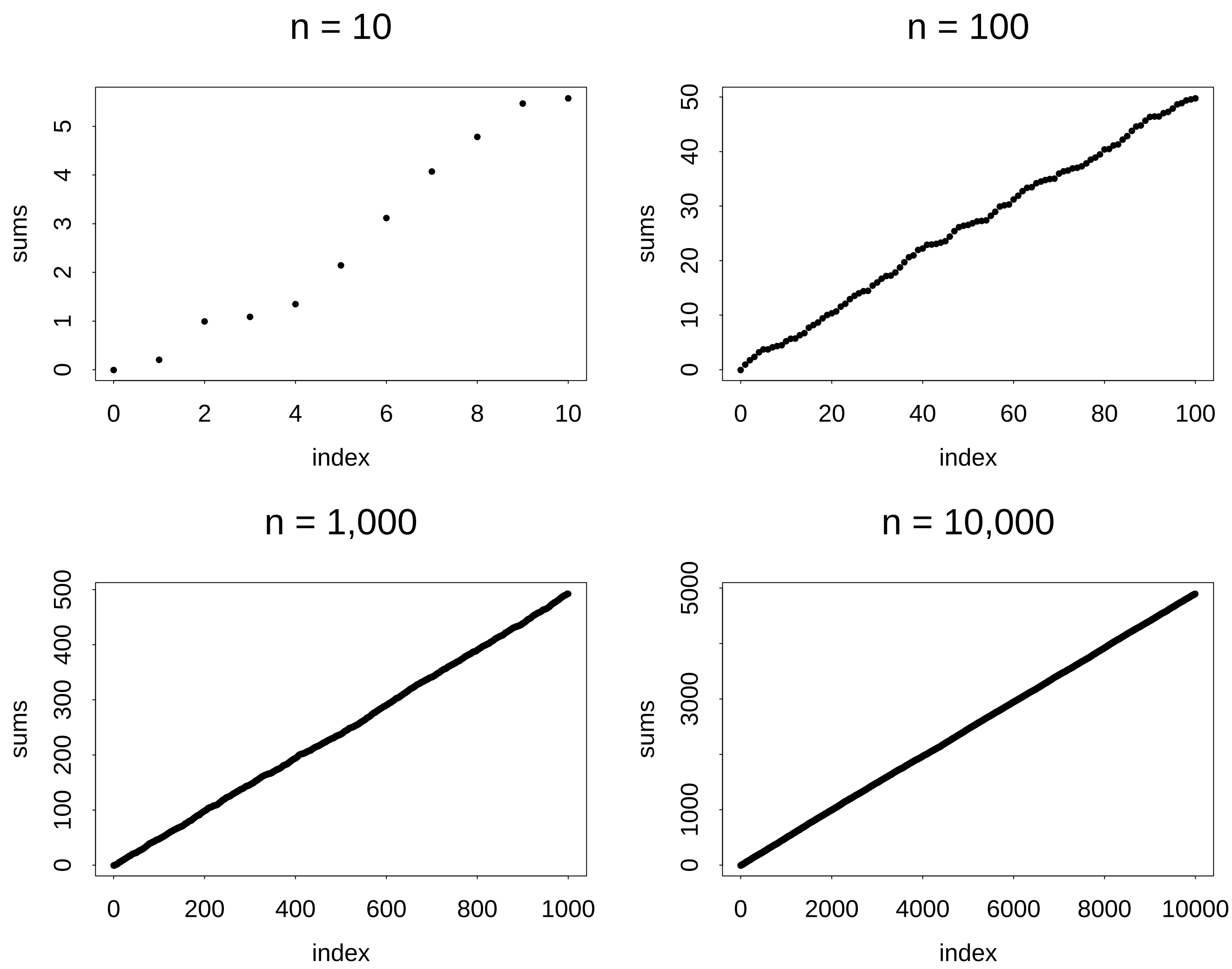
What should we see?


S_n
5
4
3
2
1
0
0
2
4
6
8
10
time index n

Look at larger sample sizes!

What should we see?

Looking at $S_1, \ldots, S_n$
when $n = 10^j$ for $j = 1, 2, 3, 4$



Plots for $n = 10^j$ with $j = 1, 2, 3, 4$

How does plotting work?

Map Into the Unit Square

(if we ignore the units on the axes)

The Plot Function: Step 1

Fit horizontally: create a function on $[0, 1]$.

For $y_0, y_1, \ldots, y_n$ given, let $x : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$x(t) \equiv y_{\lfloor nt \rfloor}, \quad 0 \leq t \leq 1,$$

where $\lfloor z \rfloor$ is the greatest integer less than or equal to z.

The Plot Function: Step 2

Fit vertically.

Place between infimum and supremum.

Fit Vertically

For $x : [0,1] \to \mathbb{R}$ given,

$$\textbf{plot}(x) \equiv (x - \textbf{inf}(x))/\textbf{range}(x),$$

where

$$\textbf{inf}(x) \equiv \inf\{x(t) : 0 \le t \le 1\}$$

$$\textbf{sup}(x) \equiv \sup\{x(t) : 0 \le t \le 1\}$$

$$\textbf{range}(x) \equiv \textbf{sup}(x) - \textbf{inf}(x)$$

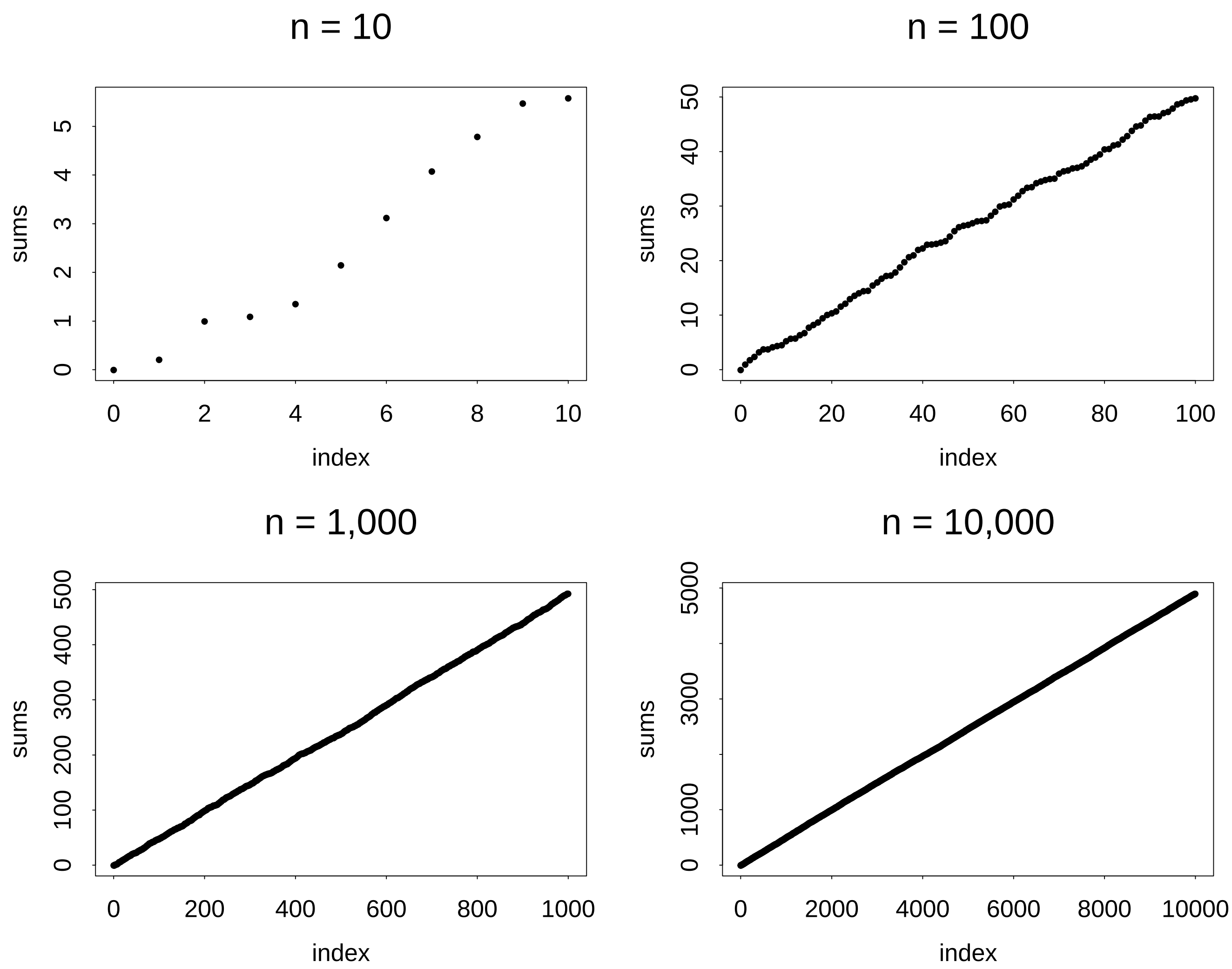
When you plot a random walk,

You get a random plot.

You get a random function.

a random function mapping $[0, 1]$ into $[0, 1]$.

You get a stochastic process.



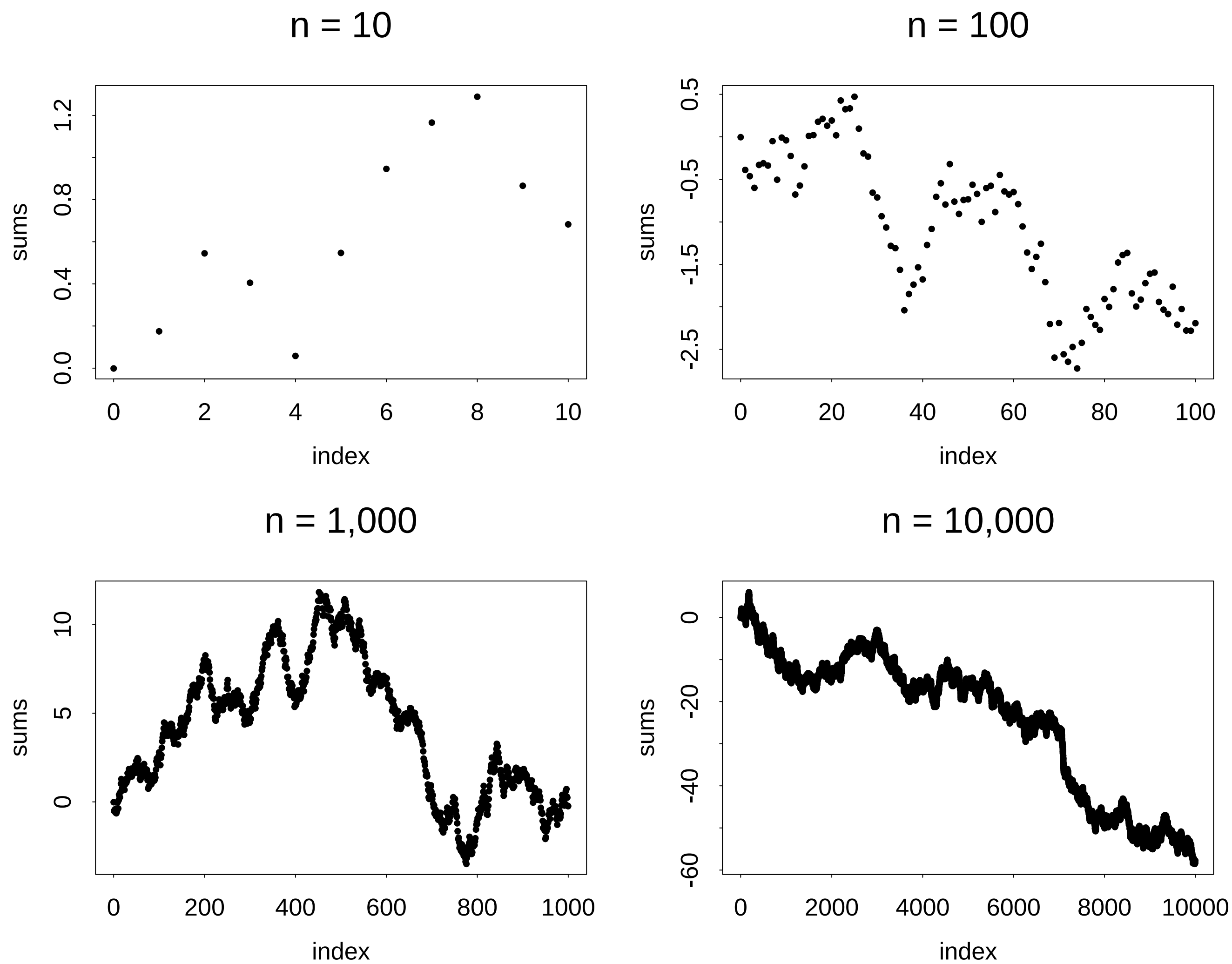
Plots for $n = 10^j$ with $j = 1, 2, 3, 4$

Modified Experiment

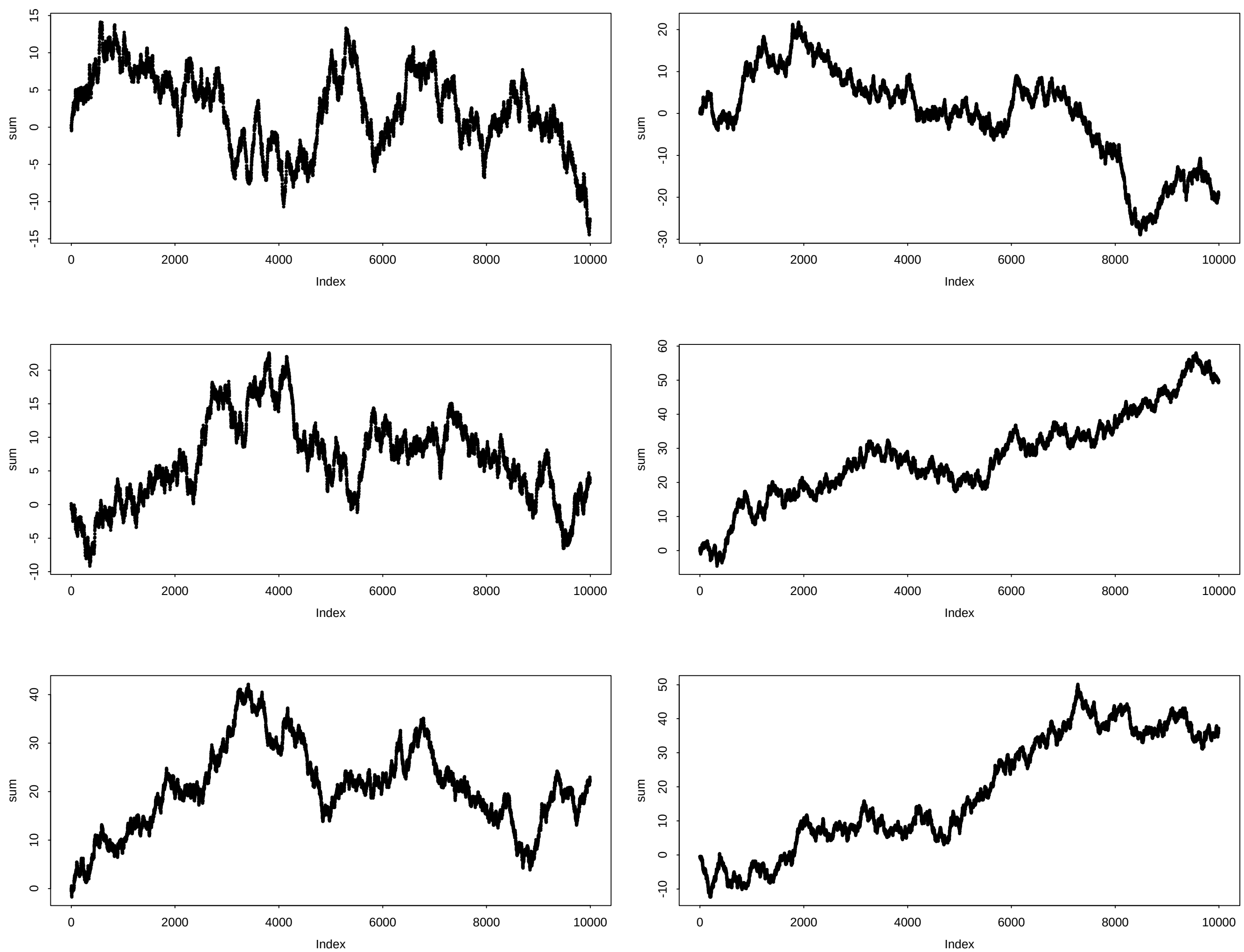
Let $X_i = U_i - 0.5$.

Construct centered random walk.

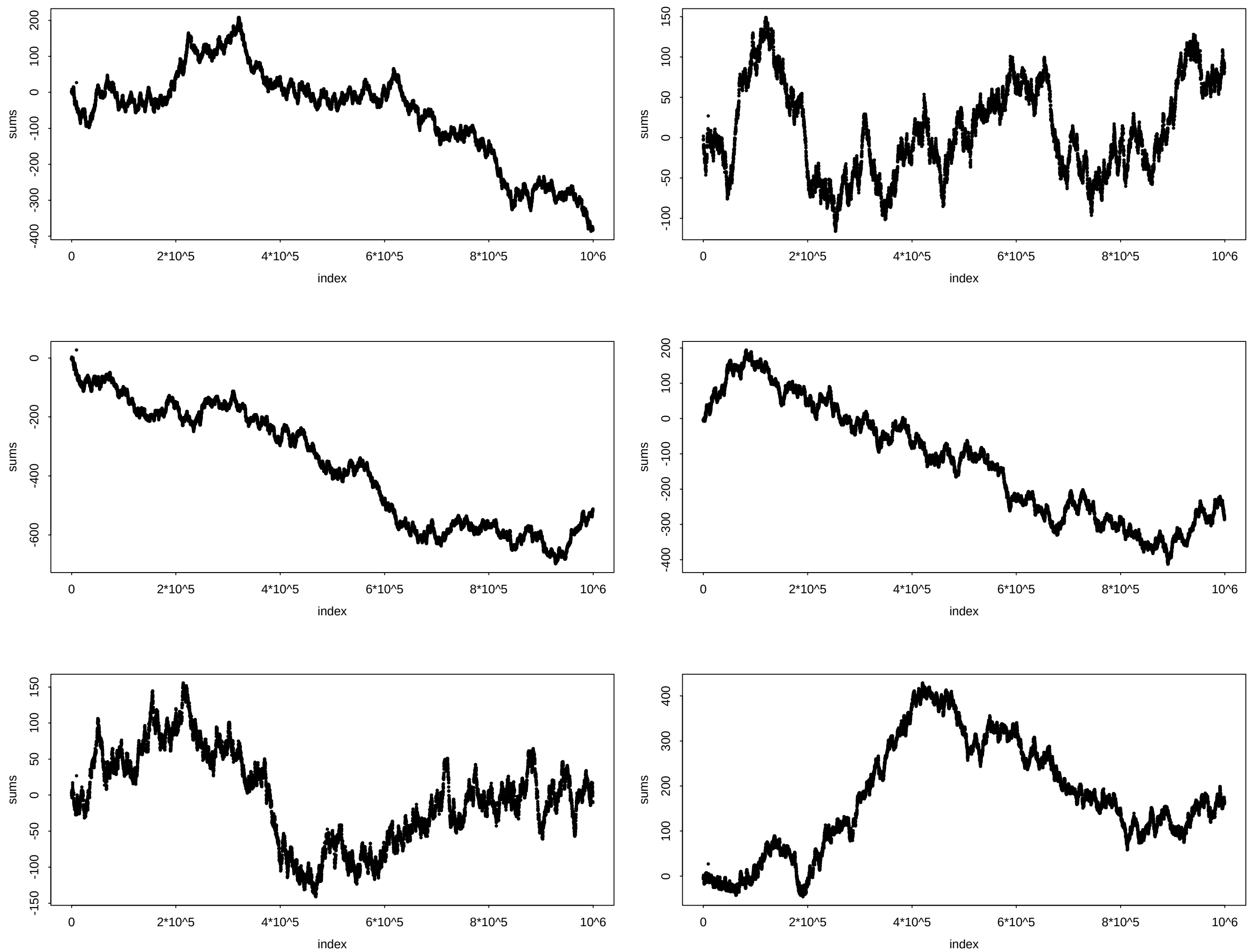
What should we see now?



The centered random walk for $n = 10^j$ with $j = 1, 2, 3, 4$



six cases for $n = 10^4$

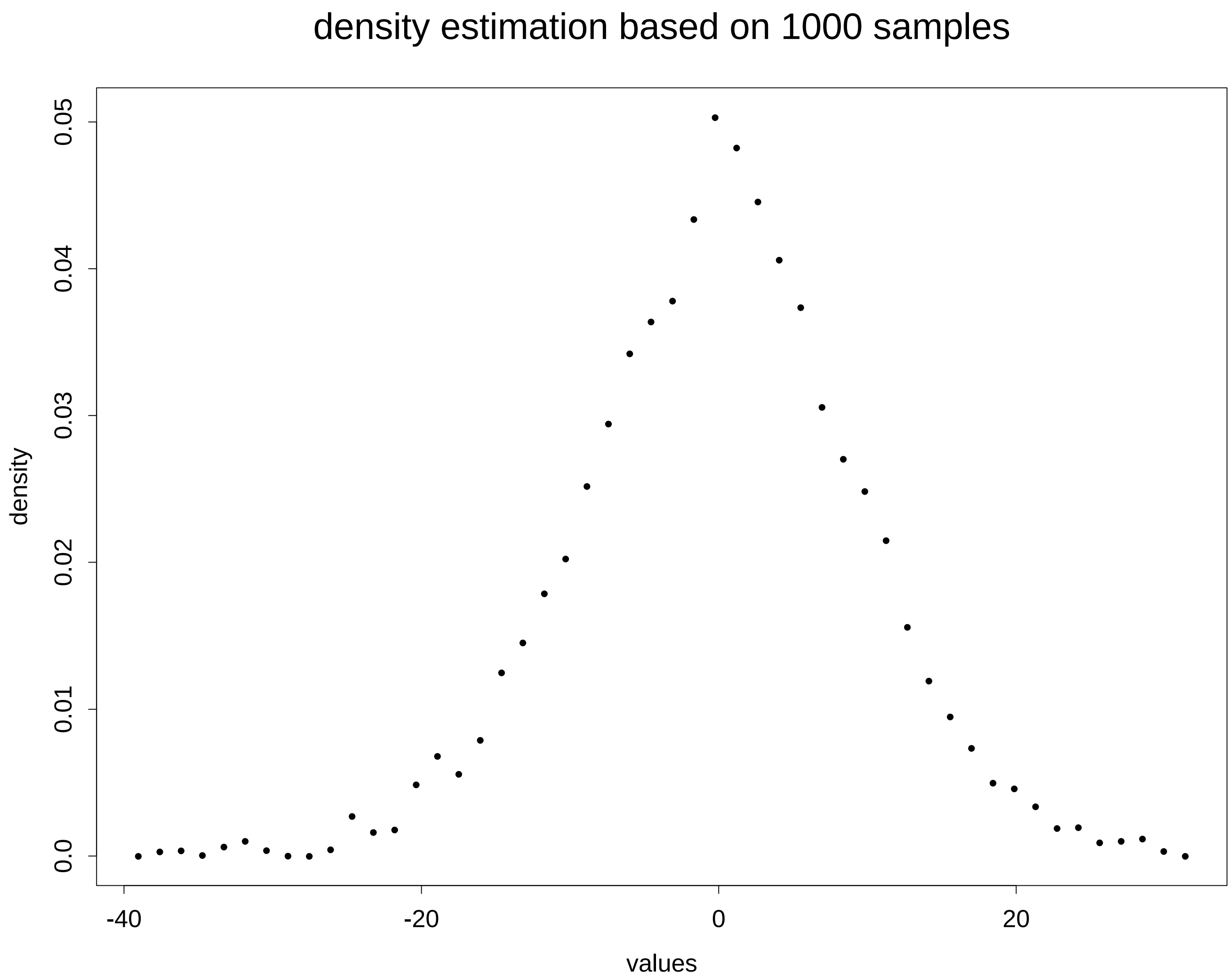


six cases for $n = 10^6$

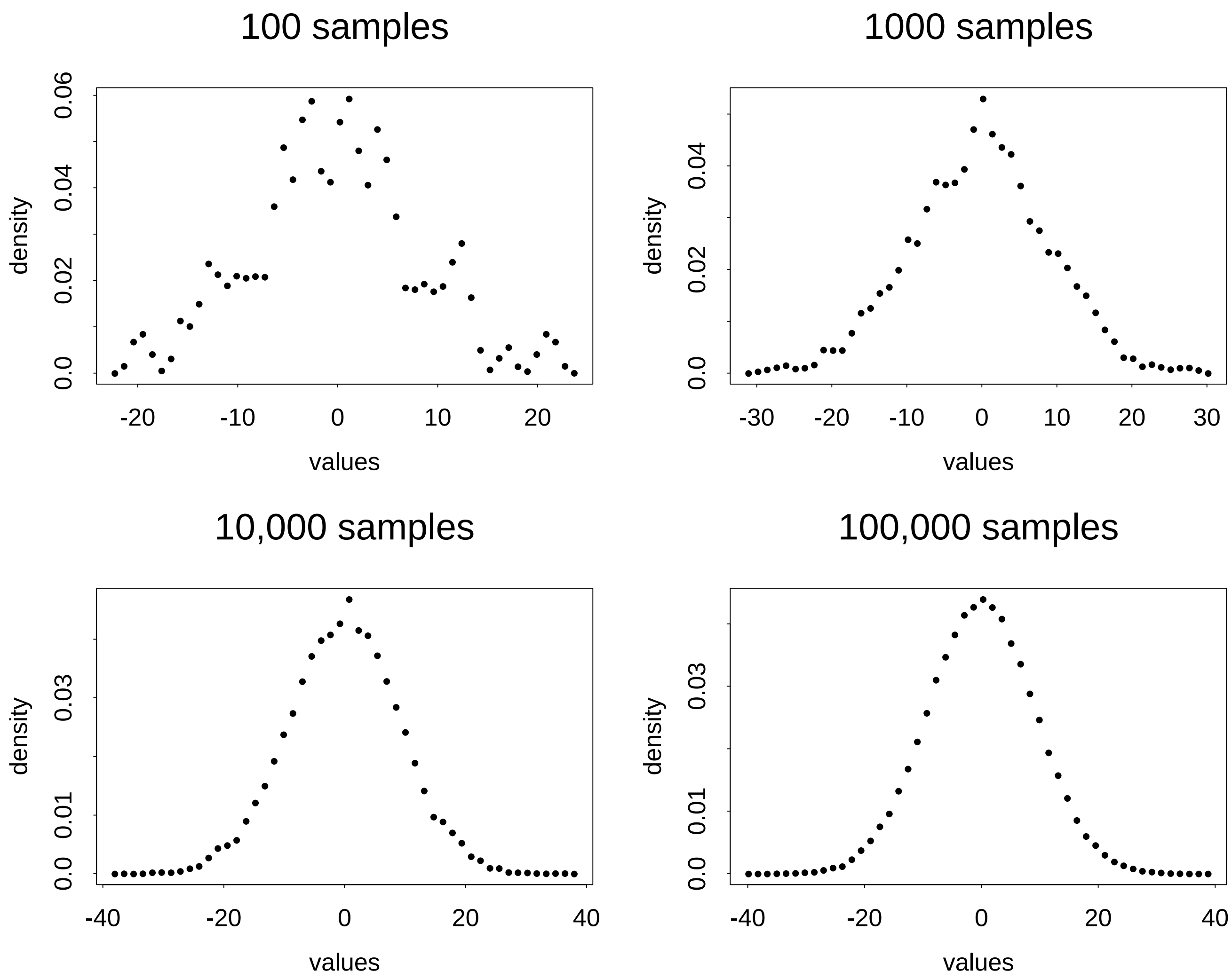
You see Brownian motion!!!

Density Estimates

for last partial sum S_n



density estimate for $S_n - mn$ with uniformly distributed summands



density estimates for $S_n - mn$ with uniformly distributed summands

The Random Plot Limit

Theorem. If the stochastic-process limit $\mathcal{S}_n \Rightarrow \mathcal{S}$ holds, where

$$\mathcal{S}_n(t) \equiv (S_{\lfloor nt \rfloor} - m\lfloor nt \rfloor)/c_n, \quad 0 \le t \le 1,$$

for some constants m and $c_n : n \ge 1$, and

$$P(range(\mathcal{S}) = 0) = 0,$$

then

$$\textbf{plot}(S_k - mk : 0 \le k \le n) \Rightarrow \textbf{plot}(\mathcal{S}).$$

Invariance Principles

New Random Steps

Let $$\begin{aligned} Y_i &= f(U_i). \\ X_i &= Y_i - EY_i. \end{aligned}$$

new centered random walk.

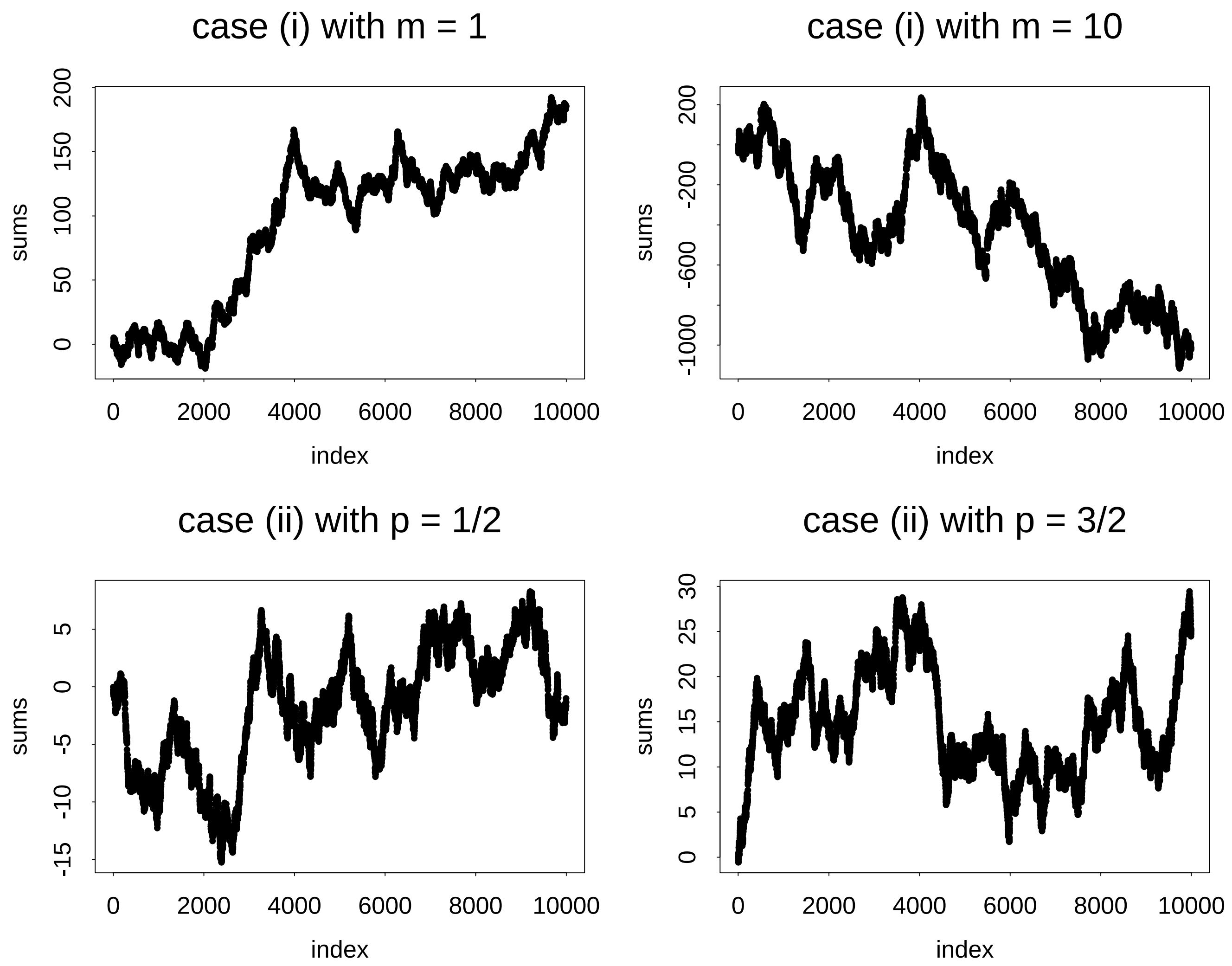
What should we see now?

Three Cases

(i) $f(U) = -m \log(U)$ for $m = 1,\ 10$

(ii) $f(U) = U^p$ for $p = 1/2,\ 3/2$

(iii) $f(U) = U^{-1/p}$ for $p = 1/2,\ 3/2$



Cases (i) and (ii) for $n = 10^4$

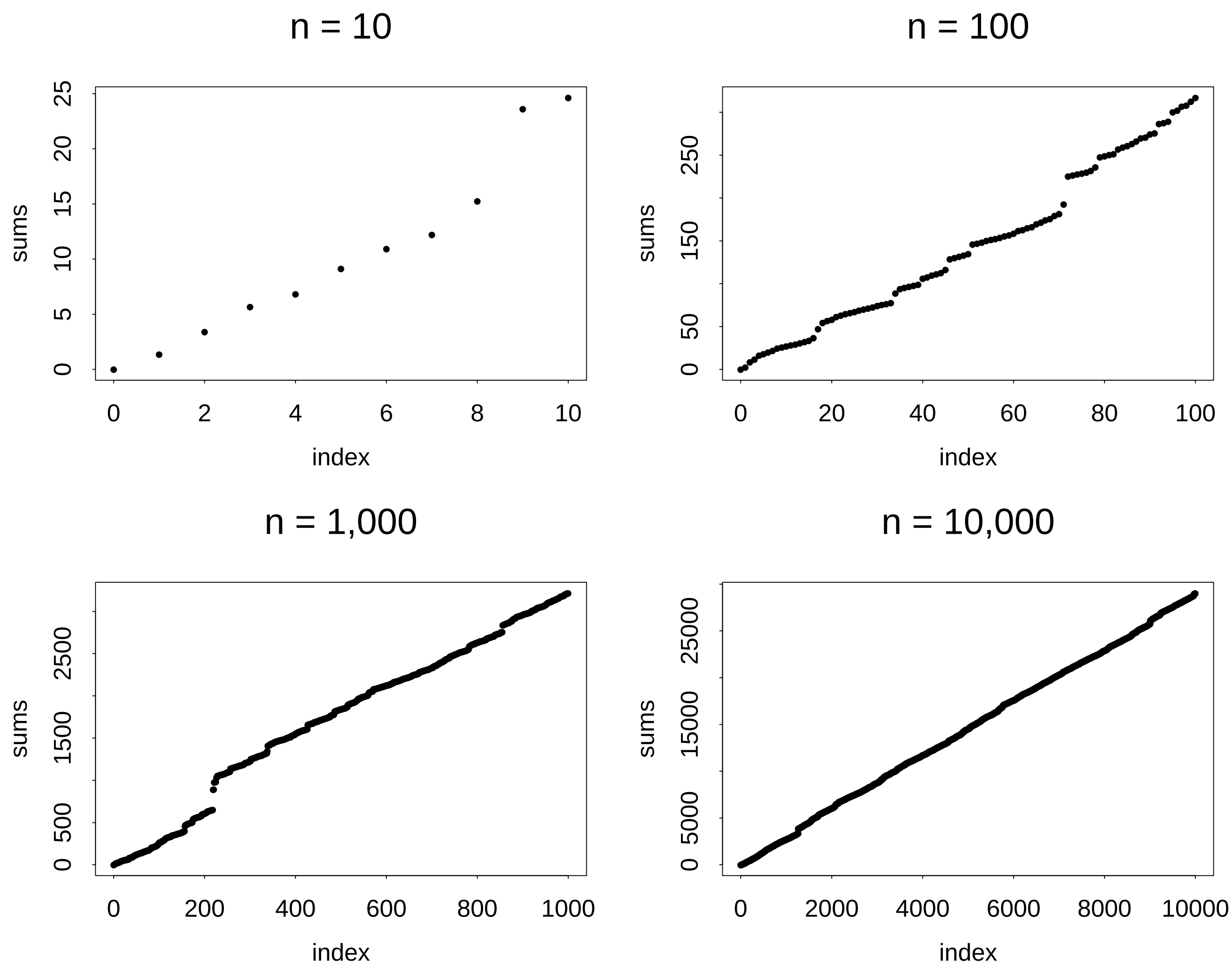
The Exception Makes the Rule

Case (iii)

(i) $f(U) = -m\log(U)$ for $m = 1,\ 10$

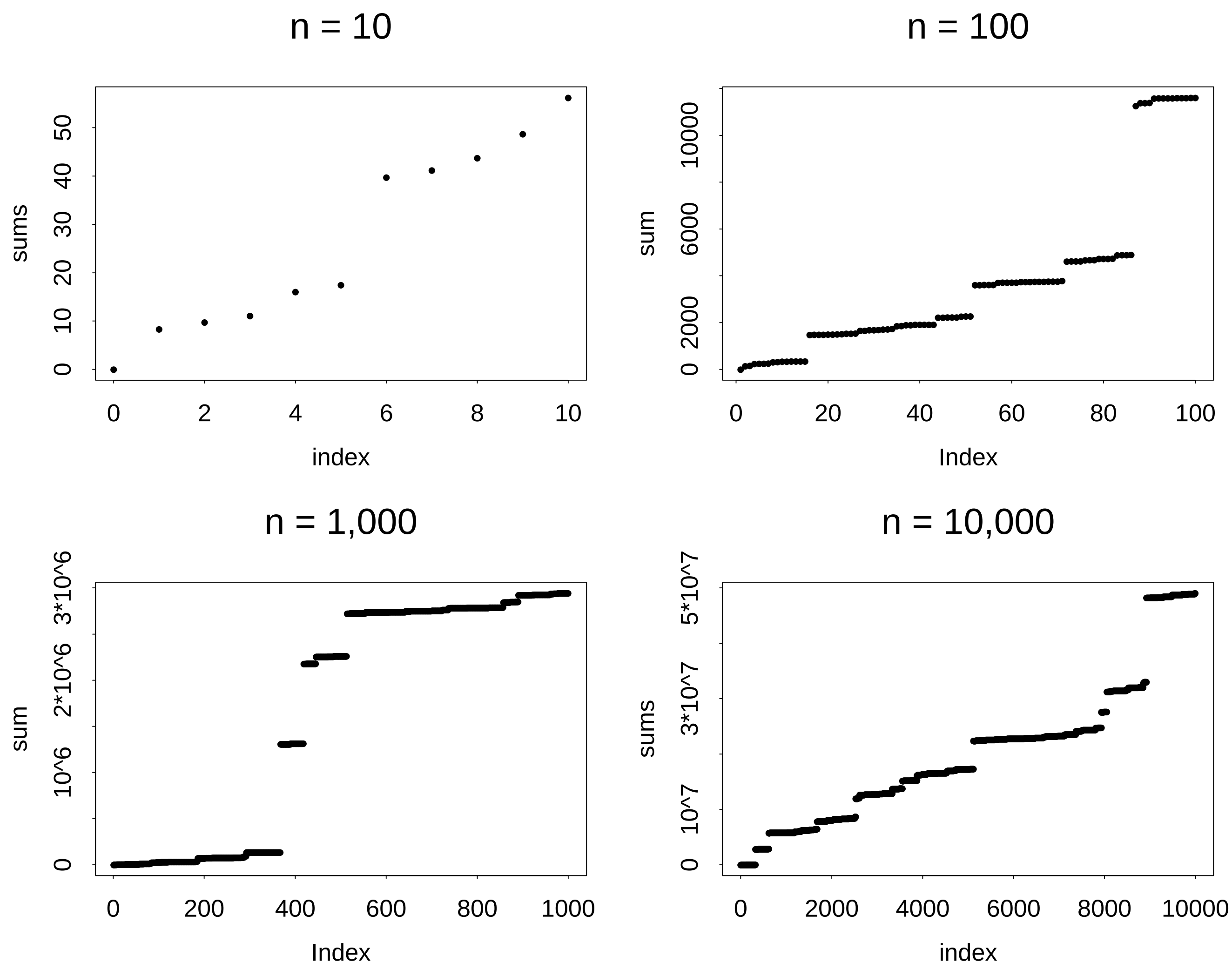
(ii) $f(U) = U^p$ for $p = 1/2,\ 3/2$

(iii) $f(U) = U^{-1/p}$ for $p = 1/2,\ 3/2$



Plots of the uncentered random walk

for $U^{-1/p}$ **with** $p = 3/2$



Plots of the uncentered random walk

for $U^{-1/p}$ **with** $p = 1/2$

Heavy Tails

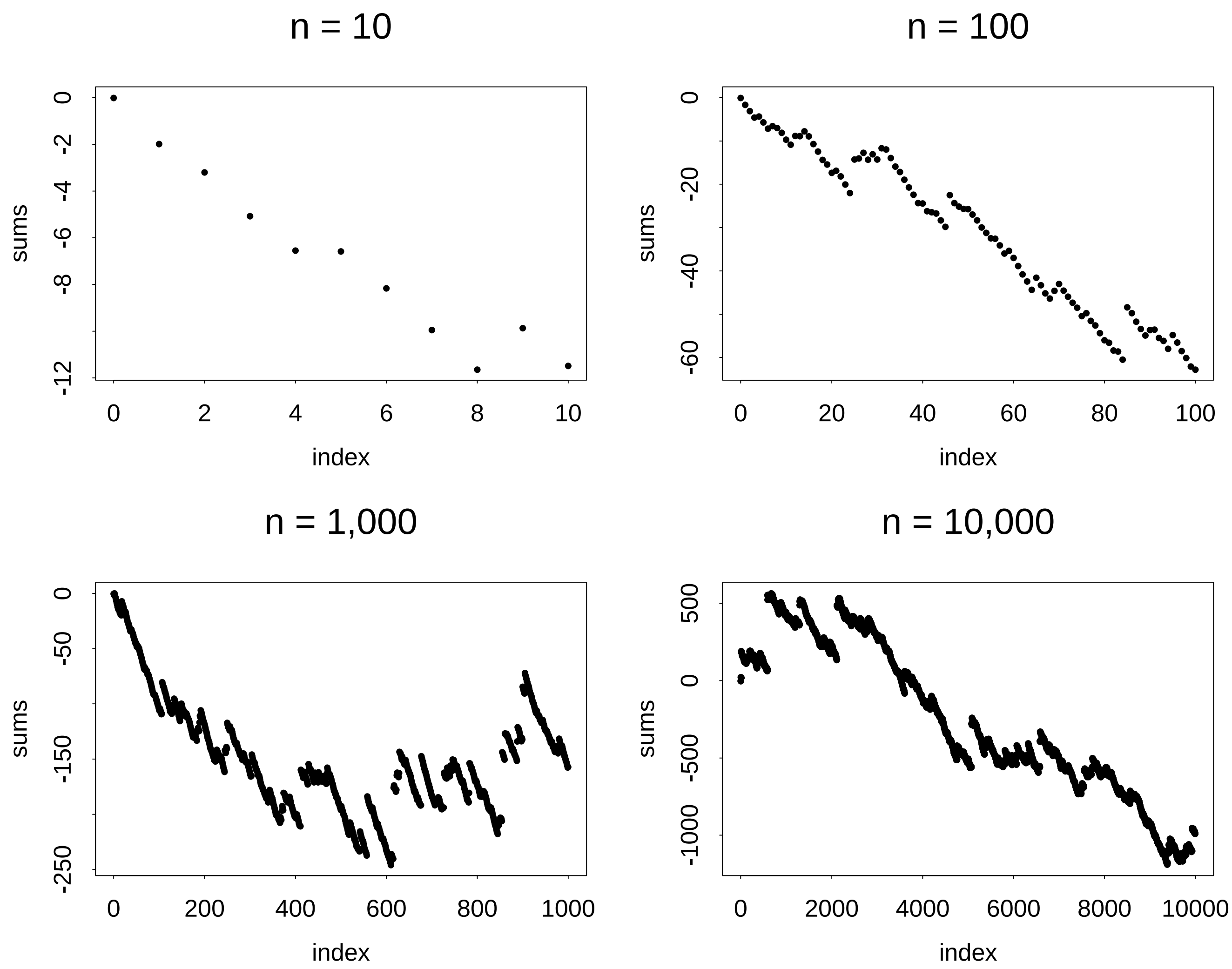
$$
\begin{aligned}
P(U^{-1/p} > t) &= P(U^{1/p} < t^{-1}). \\
&= P(U < t^{-p}) = t^{-p}
\end{aligned}
$$

Has **infinite mean** for $0 < p \leq 1$

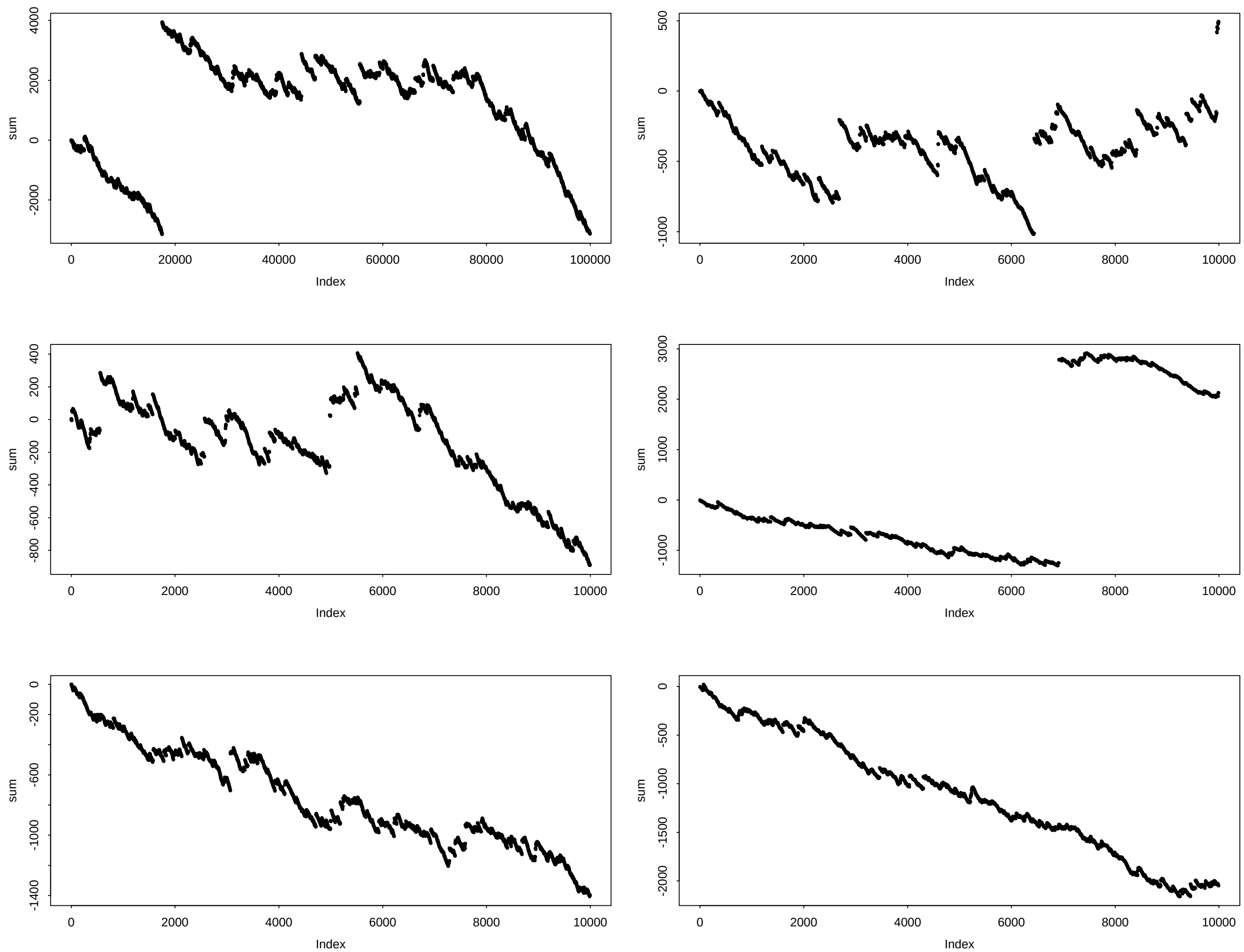
Has **infinite variance** for $0 < p \leq 2$

Plots of the Centered Random Walk

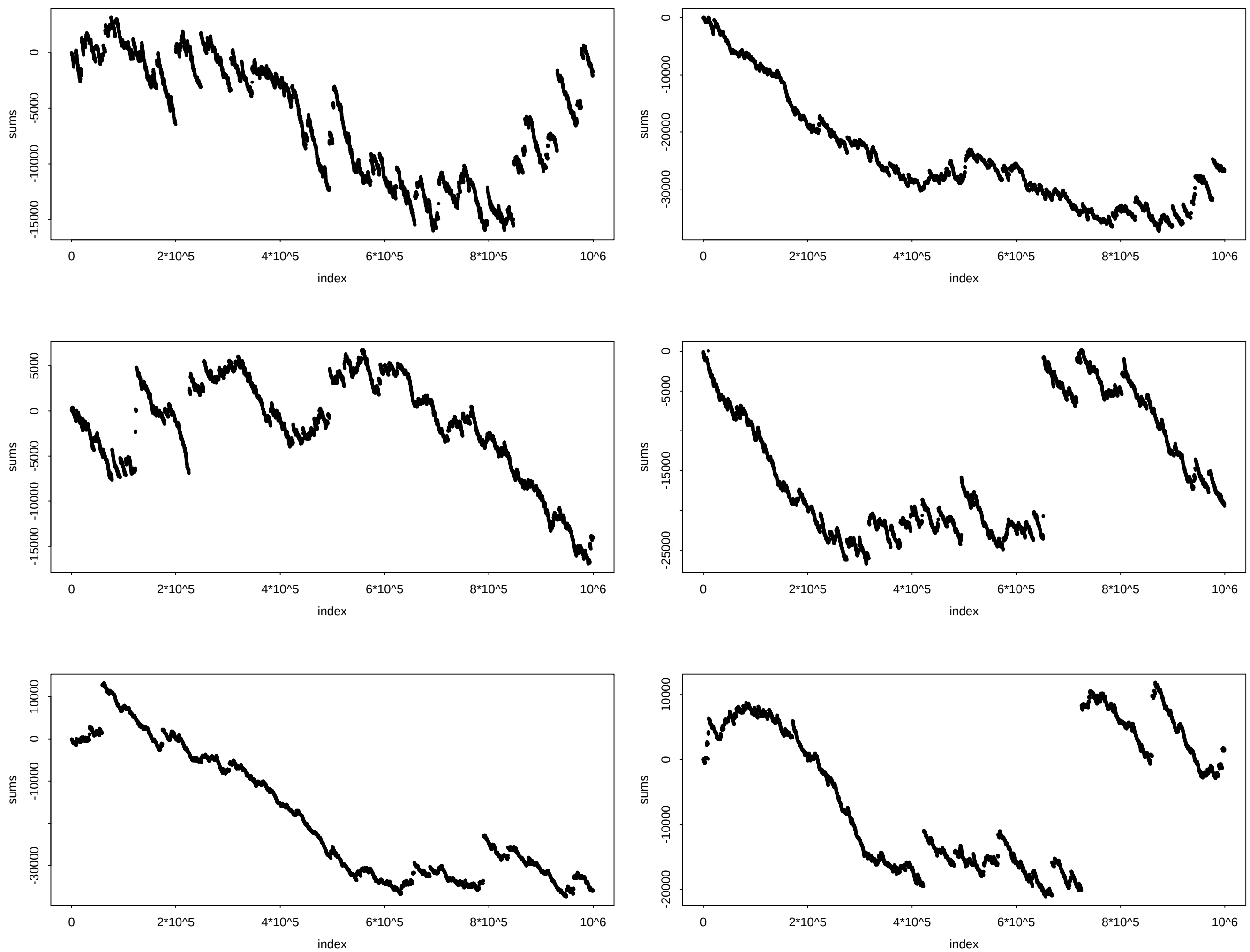
for $U^{-1/p}$ **with** $p = 3/2$



Plots of the centered random walk for $U^{-1/p}$ with $p = 3/2$



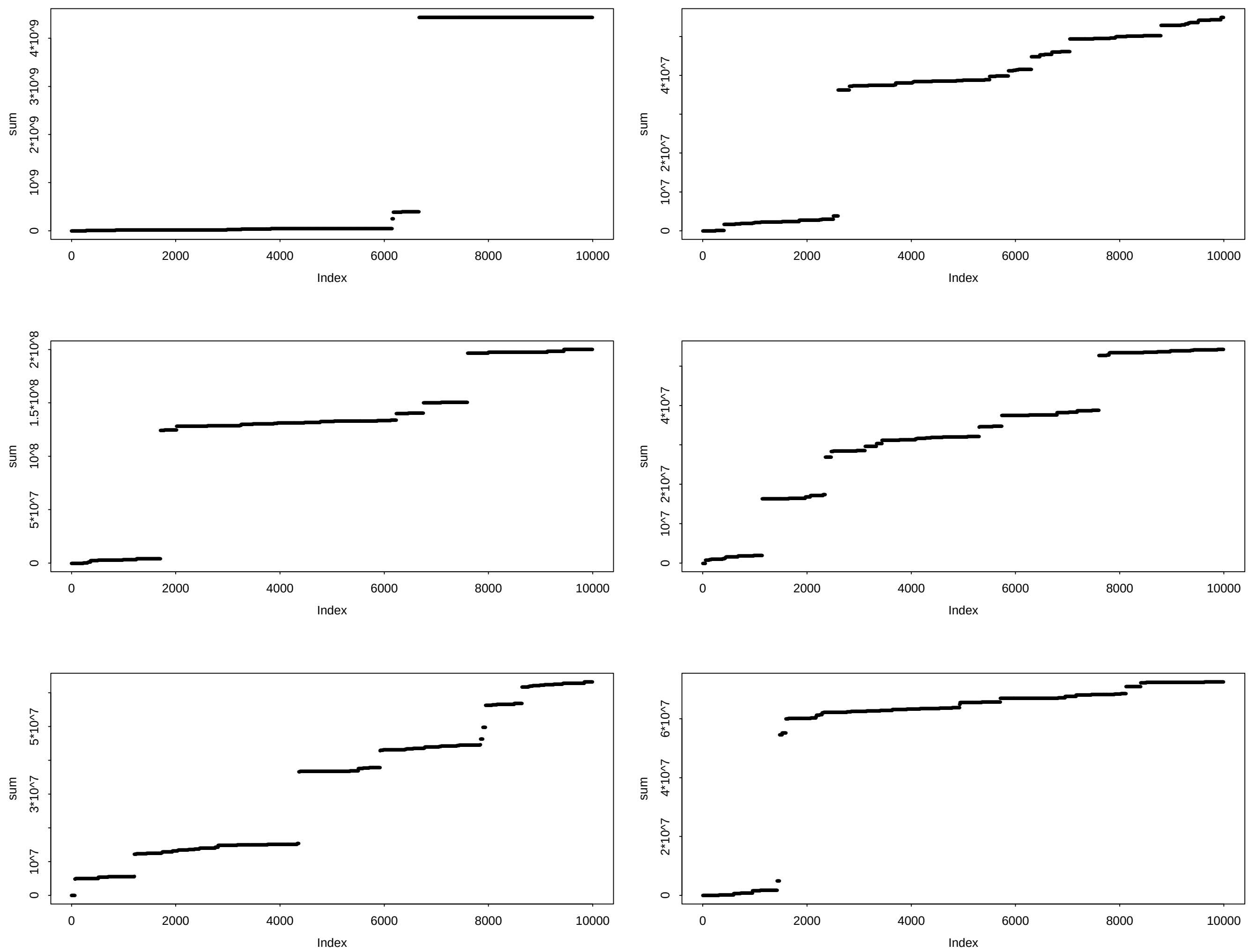
six cases for $n = 10^4$



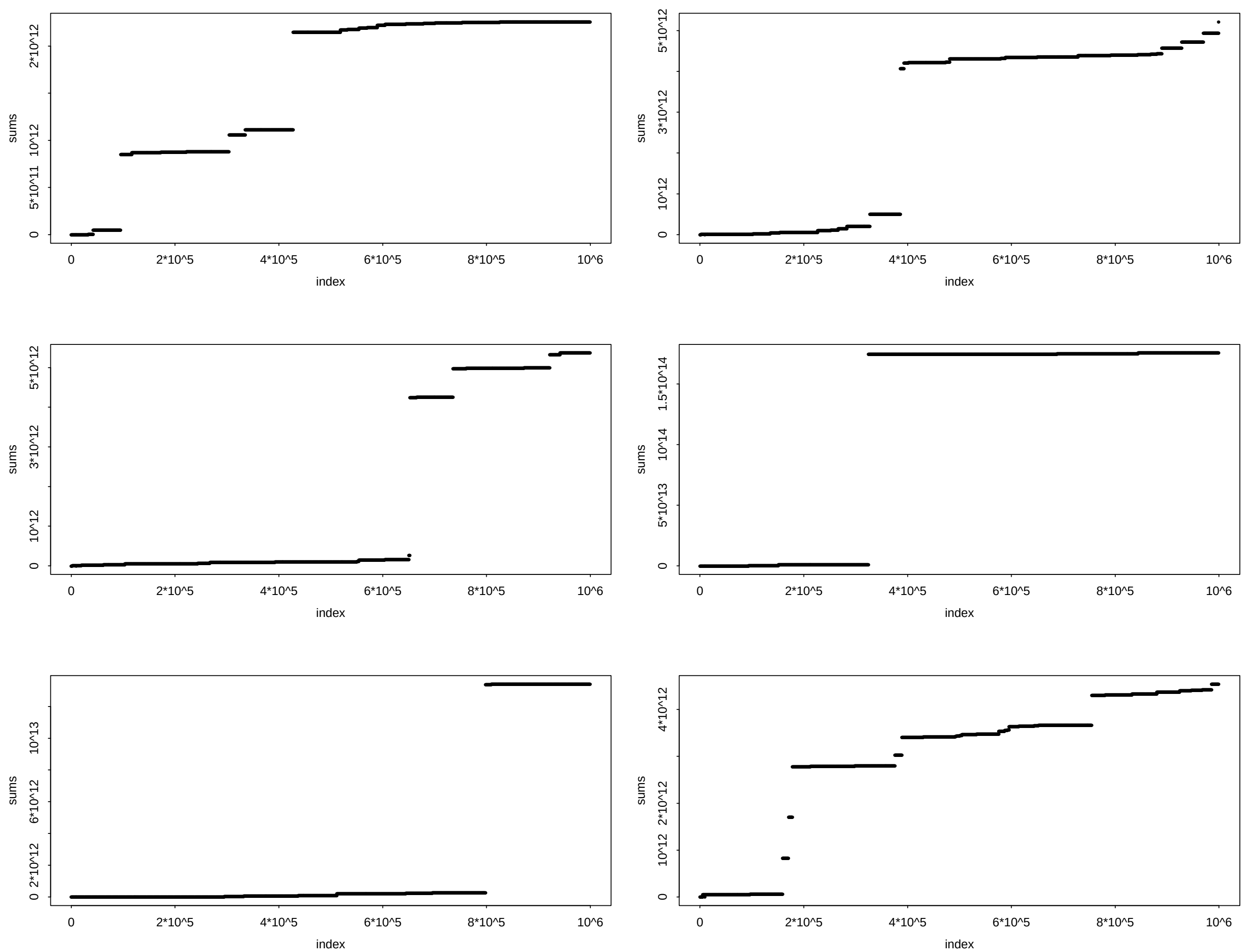
six cases for $n = 10^6$

More Plots of the Uncentered Random Walk

for $U^{-1/p}$ with $p = 1/2$



six cases for $n = 10^4$

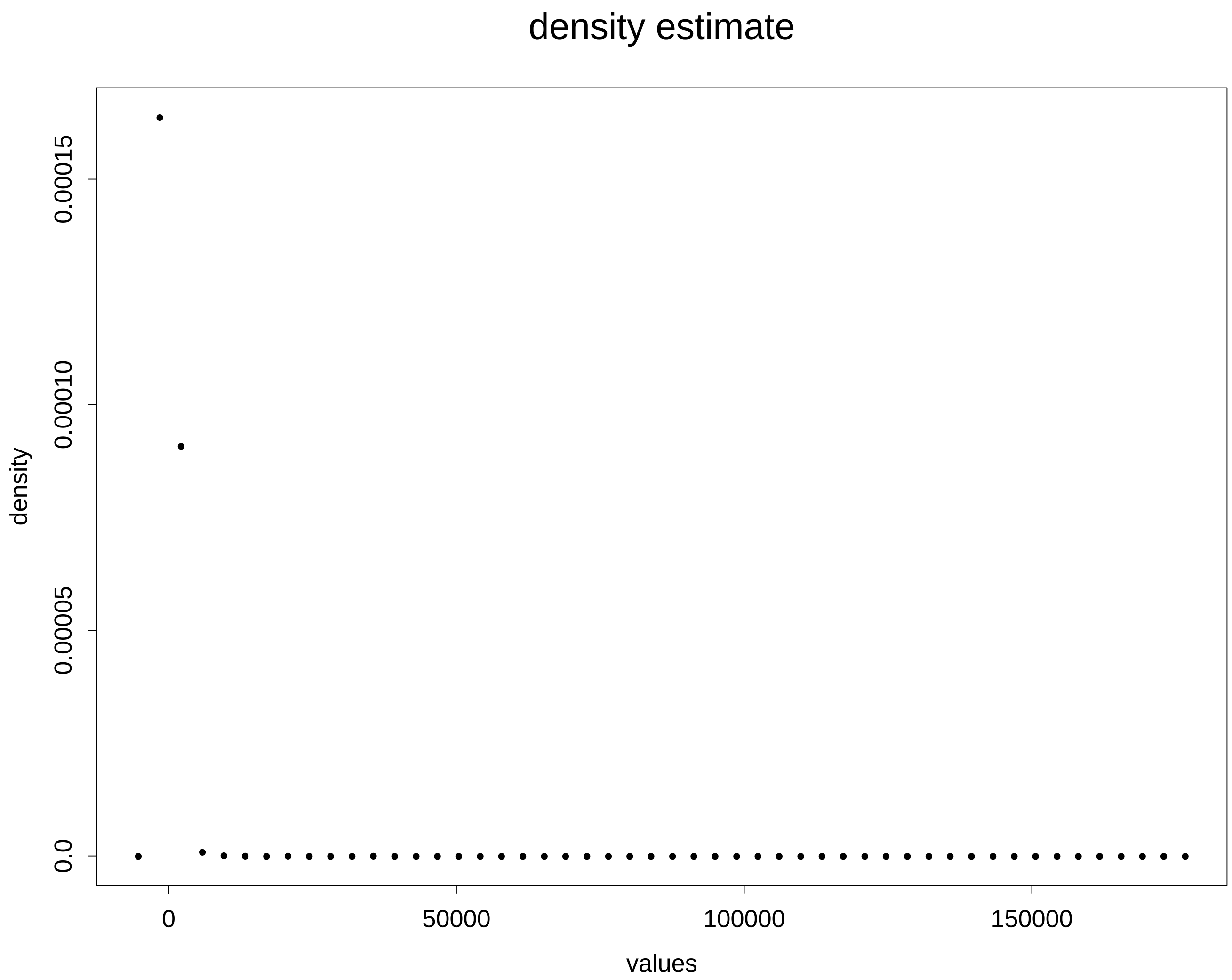


six cases for $n = 10^6$

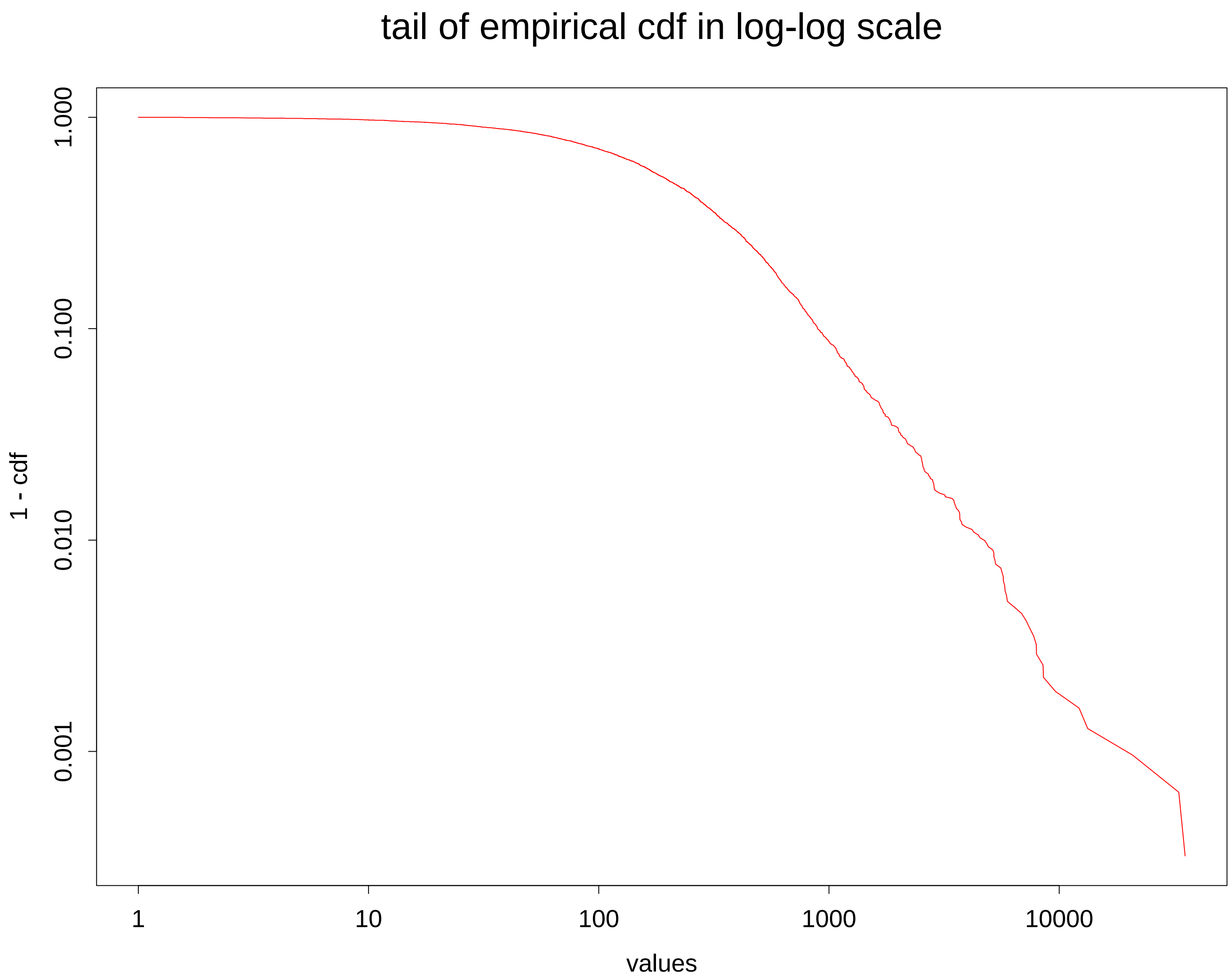
Conclusions

Plotting reveals statistical regularity.

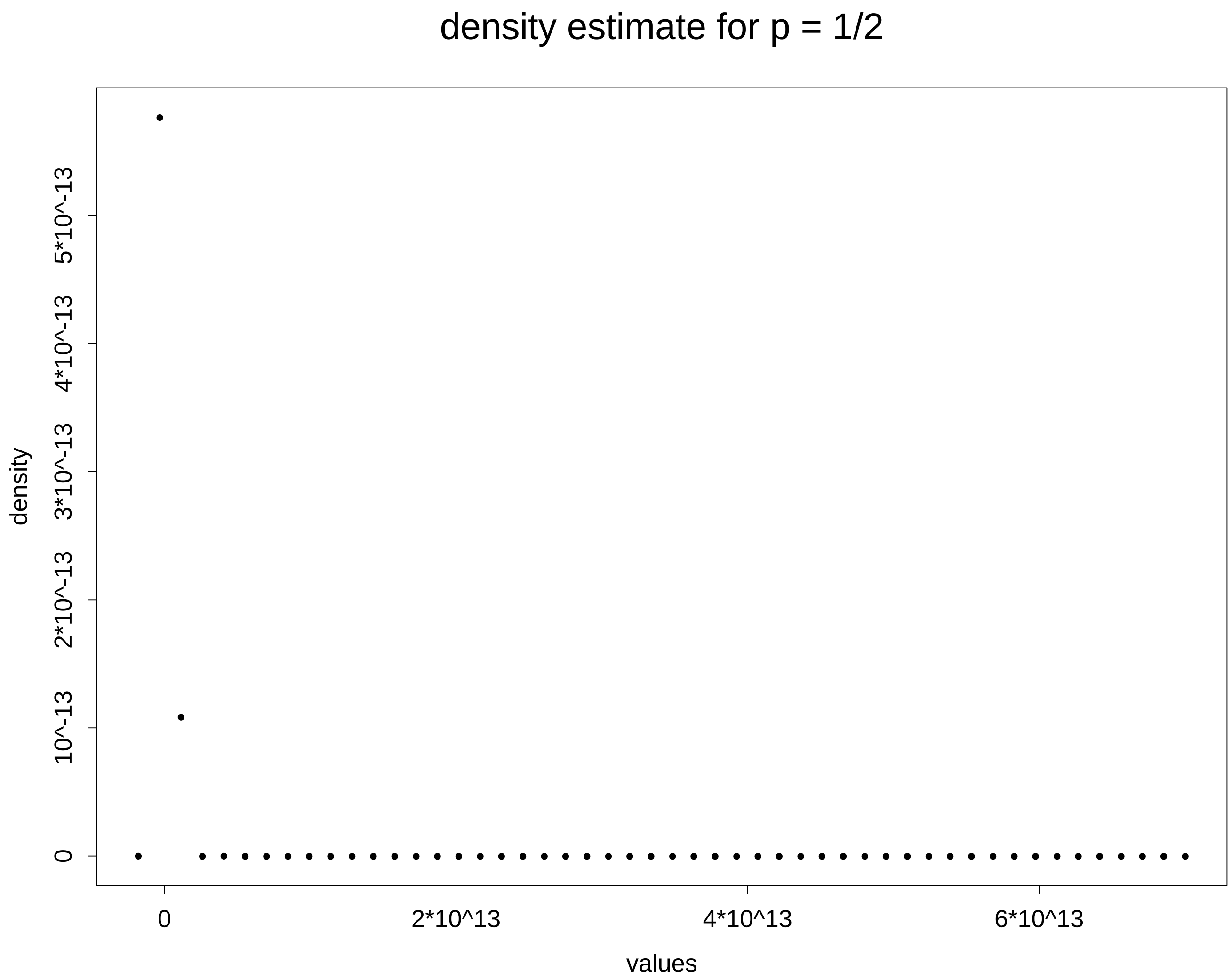
Stochastic-process limits explain the statistical regularity.



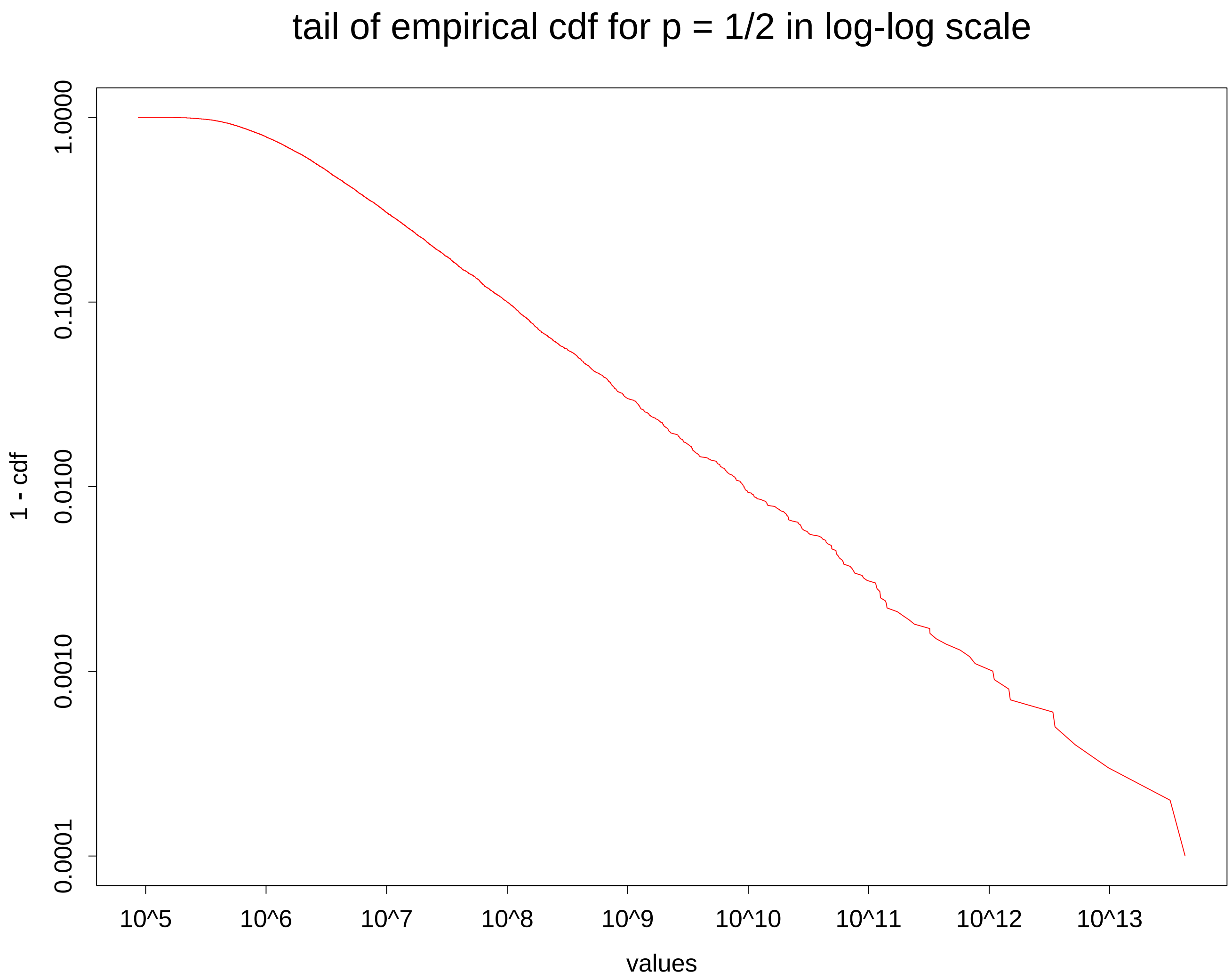
density estimate for $S_n - mn$ with summands $U^{-1/p}$ for $p = 3/2$



tail of empirical cdf of $S_n - mn$ with summands $U^{-1/p}$ for $p = 3/2$



density estimate for $S_n - mn$ with summands $U^{-1/p}$ for $p = 1/2$



tail of empirical cdf of $S_n - mn$ with summands $U^{-1/p}$ for $p = 1/2$