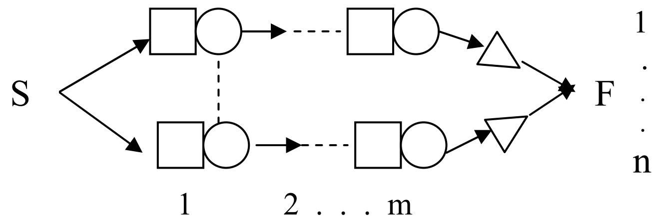
Empirical + Theoretical Approximation

n jobs, in parallel

m tasks per job, with intermediate queues

$\Rightarrow$ $n \times m$ resources



λ arrival rate

$1/\mu$ service time (avg)

$\Rightarrow$ $\rho = \lambda / \mu$

T = average sojourn time for a project

$$\frac{T}{1/\mu} \approx \frac{\mathbf{m}}{1-\rho}\left(1+\frac{\mathbf{ln\,n}}{\mathrm{m}} \cdot \frac{\ln {}^{m}\!/\!_{\rho}}{2}\right)$$

within 5% of (many) simulations

Resource constraints vs. Synch. Gaps

m ln n

? Why

Why logs?

$$T_i \sim \text{iid} \quad \exp(\mu), \qquad i = 1,\ldots,n$$

Know: $\min\limits_{1\le i\le n} T_i \sim \exp(n\mu)$ via "exp competition"

$\max\limits_{1\le i\le n} T_i \sim ?$

$$\max_{1\le i\le n} T_i \overset{d}{=} \min_{1\le i\le n} T_i \underbrace{\text{“}+\text{”}\ \min_{1\le i\le n-1} T_i + \cdots}_{\text{independent}} + T_1$$

$$E\max = \frac{1}{n\mu} + \frac{1}{(n-1)\mu} + \cdots + \frac{1}{\mu} = \frac{1}{\mu}\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right]$$

$$\approx \frac{1}{\mu}[\ln n + \gamma]$$

Euler constant 0.57722

$$E \max_{1\le i\le n} T_i = E(T_1)\sum_{i=1}^{n}\frac{1}{i} \approx E(T_1)\ (\ln n + \gamma)$$

But more! Using $\mathrm{Var}\, T = (ET)^2$ for $T \sim \exp$,

$$\mathrm{Var} \max_{1\le i\le n} T_i = \frac{1}{\mu^2}\left[1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right] \approx \mathrm{Var}\, T_1 \cdot \frac{\pi^2}{6}$$

Conclude: ($\mu = 1$ for simplicity)

$$\left.\begin{array}{l} E \approx \ln n \\ \sigma \approx \pi/\sqrt{6} \end{array}\right\} \Rightarrow \text{max is essentially constant, for n large.}$$